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Abstract

We establish deep and remarkable connections among partial differential equations (PDEs) integrable by different methods: the inverse spectral transform method, the method of characteristics and the Hopf–Cole transformation. More concretely, (1) we show that the integrability properties (Lax pair, infinitely-many commuting symmetries, large classes of analytic solutions) of (2+1)-dimensional PDEs integrable by the inverse scattering transform method (*S*-integrable) can be generated by the integrability properties of the (1+1)-dimensional matrix Bürgers hierarchy, integrable by the matrix Hopf–Cole transformation (*C*-integrable). (2) We show that the integrability properties (i) of *S*-integrable PDEs in (1+1) dimensions, (ii) of the multidimensional generalizations of the $GL(M, \mathbb{C})$ self-dual Yang–Mills equations and (iii) of the multidimensional Calogero equations can be generated by the integrability properties of a recently introduced multidimensional matrix equation solvable by the method of characteristics. To establish the above links, we consider a block Frobenius matrix reduction of the relevant matrix fields, leading to integrable chains of matrix equations for the blocks of such a Frobenius matrix, followed by a systematic elimination procedure of some of these blocks. The construction of large classes of solutions of the soliton equations from solutions of the matrix Bürgers hierarchy turns out to be intimately related to the construction of solutions in Sato theory. (3) Finally, we show that suitable generalizations of the block Frobenius matrix reduction of

the matrix Bürgers hierarchy generates PDEs exhibiting integrability properties in common with both *S*- and *C*-integrable equations.

1. Introduction

Integrable nonlinear partial differential equations (PDEs) can be grouped into different classes, depending on their method of solution. We distinguish the following three basic classes.

(1) Equations solvable by the method of characteristics [1], hereafter called, for the sake of brevity, *Ch*-integrable, such as the following matrix PDE in arbitrary dimensions [2]:

$$w_t + \sum_{i=1}^{N} w_{x_i} \rho^{(i)}(w) + [B, w]\sigma(w) = 0, \tag{1}$$

where w is a square matrix and $\rho^{(i)}(\cdot)$, $\sigma(\cdot)$ are scalar functions representable as positive power series, or such as the vector equations solvable by the generalized hodograph method [3–8].

(2) Equations integrable by a simple change of variables, often called *C*-integrable [9], such as the matrix Bürgers equation [10]

$$w_t - Bw_{xx} - 2Bw_x w + [w, B](w_x + w^2) = 0, \tag{2}$$

where B is any constant square matrix, linearizable by the matrix version of the Hopf–Cole transformation $\Psi_x = w\Psi$ [11].

(3) Equations integrable by less elementary methods of spectral nature, the inverse spectral transform (IST) [12–16] and the dressing method [16–20], often called *S*-integrable [9] or soliton equations. Within this class of equations, we distinguish four different subclasses, depending on the nature of the associated spectral theory.

 (a) Soliton equations in (1+1) dimensions such as, for instance, the Korteweg–de Vries (KdV) [12, 21] and the nonlinear Schrödinger (NLS) [22] equations, whose inverse problems are local Riemann–Hilbert (RH) problems [13, 15].

 (b) Their (2+1)-dimensional generalizations, such as the Kadomtsev–Petviashvily (KP) [23] and Davey–Stewartson (DS) [24] equations, whose inverse problems are nonlocal RH [25, 26] or $\bar{\partial}$-problems [27].

 (c) The self-dual Yang–Mills (SDYM) equation [28, 29] and its generalizations in arbitrary dimensions.

 (d) Multidimensional PDEs associated with one-parameter families of commuting vector fields, whose novel IST, recently constructed in [30, 31], is characterized by nonlinear RH [30, 31] or $\bar{\partial}$ [32] problems. Distinguished examples are the dispersion-less KP equation, the heavenly equation of Plebanski [33] and the following integrable system of PDEs in $(N + 4)$ dimensions [30]:

$$\vec{v}_{t_1 z_2} - \vec{v}_{t_2 z_1} + \sum_{i=1}^{N} (\vec{v}_{z_1} \cdot \nabla_{\vec{x}})\vec{v}_{z_2} - \sum_{i=1}^{N} \vec{v}_{z_2} \cdot \nabla_{\vec{x}}\vec{v}_{z_1} = \vec{0}, \tag{3}$$

where $\vec{v}$ is an *N*-dimensional vector and $\nabla_{\vec{x}} = (\partial_{x_1}, \ldots, \partial_{x_N})$.

Each one of the above methods of solution allows one to solve a particular class of PDEs and is not applicable to other classes.

Recently, several variants of the classical dressing method have been suggested, allowing to unify the integration algorithms for *C*- and *S*-integrable PDEs [34], for *C*- and *Ch*-integrable PDEs [35], and for *S*- and *Ch*-integrable PDEs [36]. In particular, the relation between the matrix PDE (1), integrable by the method of characteristics, and the $GL(M, \mathbb{C})$ SDYM equation has been recently established in [37]. As a consequence of this result, it was shown that the SDYM equation admits an infinite class of lower-dimensional reductions which are integrable by the method of characteristics.

In this paper, we extend the results of [37], showing the existence of remarkably deep relations among *S*-, *C*- and *Ch*-integrable systems. More precisely, we do the following.

(1) We show (in section 2) that the integrability properties (Lax pair, infinitely-many commuting symmetries, large classes of analytic solutions) of the *C*-integrable (1+1)-dimensional matrix Bürgers hierarchy can be used to generate the integrability properties of *S*-integrable PDEs in (2+1) dimensions, such as the *N*-wave, KP and DS equations; this result is achieved using a block Frobenius matrix reduction of the relevant matrix field of the matrix Bürgers hierarchy, leading to integrable chains of matrix equations for the blocks of such a Frobenius matrix, followed by a systematic elimination procedure of some of these blocks. The construction of large classes of solutions of the soliton equations from solutions of the matrix Bürgers hierarchy turns out to be intimately related to the construction of solutions in Sato theory [38–41]. On the way back, starting with the Lax pair eigenfunctions of the derived *S*-integrable systems, we show that the coefficients of their asymptotic expansions, for large values of the spectral parameter, coincide with the elements of the above integrable chains, obtaining an interesting spectral meaning of such chains. It follows that, compiling these coefficients into the Frobenius matrix, one constructs the *C*-integrable matrix Bürgers hierarchy and its solutions from the eigenfunctions of the *S*-integrable systems.

(2) We show (in section 3) that the integrability properties of the multidimensional matrix equation (1), solvable by the method of characteristics, can be used to generate the integrability properties of

   (a) *S*-integrable PDEs in (1+1) dimensions, such as the *N*-wave, KdV, modified KdV (mKdV) and NLS equations (in section 3.2);

   (b) *S*-integrable multidimensional generalizations of the $GL(M, \mathbb{C})$ SDYM equations (in section 3.3); this derivation from the simpler and basic matrix equation (1), allows one to uncover for free two important properties of such equations: a convenient parametrization, given in terms of the blocks of the Frobenius matrix, allowing one to reduce the number of equations by half, and the existence of a large class of solutions describing the gradient catastrophe of multidimensional waves.

   (c) *S*-integrable multidimensional Calogero equations [42–46] (in section 3.4).

   As before, these results are obtained considering a block Frobenius matrix reduction, leading to integrable chains, followed by a systematic elimination procedure of some of their elements. Vice versa, such chains are satisfied by the coefficients of the asymptotic expansion, for large values of the spectral parameter, of the eigenfunctions of the soliton equations.

(3) We show (in section 4) that a proper generalization of the block Frobenius matrix reduction of the matrix Bürgers hierarchy can be used to construct the integrability properties of nonlinear PDEs exhibiting properties in common with both *S*- and *C*-integrable equations.

Figure 1 shows the diagram summarizing the connections discussed in sections 2 and 3.
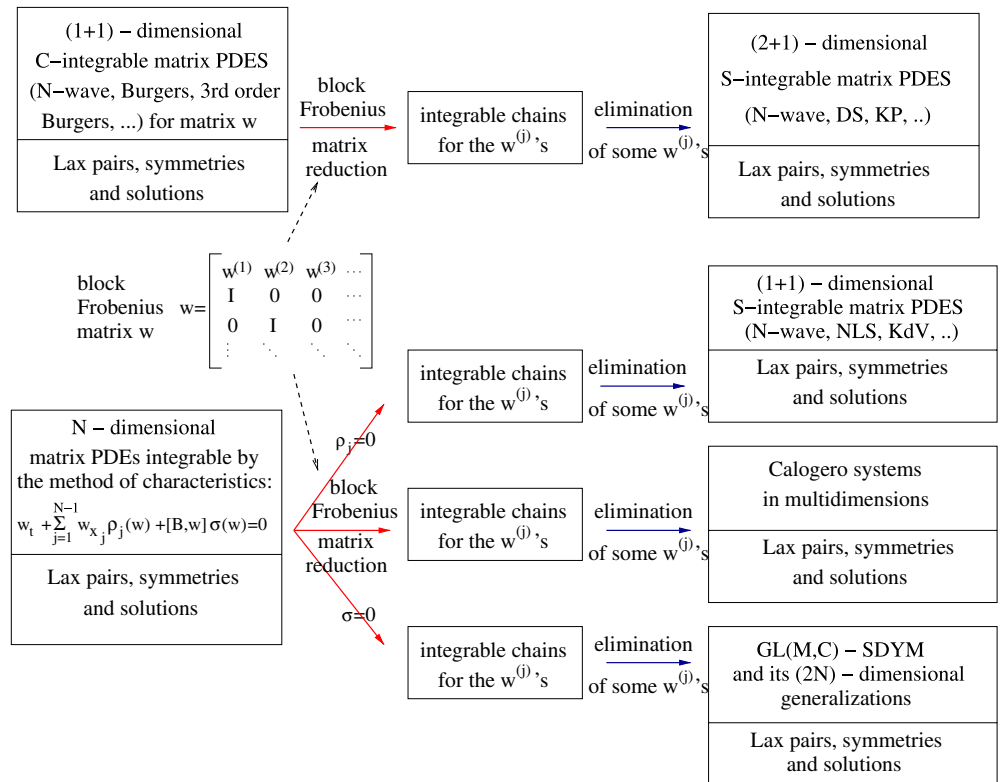
Figure 1. The remarkable relations among PDEs integrable by the inverse spectral transform method, by the method of characteristics and by the Hopf–Cole transformation.

We end this introduction mentioning previous work related to our main findings as follows. (i) The matrix Bürgers equation (2) with $B = I$, together with the block Frobenius matrix reduction (13), have been used in [47] to construct some explicit solutions of the linear Schrödinger and diffusion equations. (ii) As already mentioned, once the connections illustrated in section 2 are exploited to construct large classes of solutions of soliton equations from simpler solutions of the matrix Bürgers hierarchy, the corresponding formalism turns out to be intimately related to the construction of solutions of soliton equations in Sato theory.

2. Relation between *C*- and *S*-integrability

Usually *C*- and *S*-integrable systems are considered as completely integrable systems with different integrability features. In this section we show the remarkable relations between them.

2.1. C-integrable PDEs

It is well known that the hierarchy of *C*-integrable systems associated with the matrix Hopf–Cole transformation

$$\Psi_x = w\Psi \tag{4}$$

can be generated by the compatibility condition between equation (4) and the following hierarchies of linear commuting flows (the hierarchy generated by higher x-derivatives and its replicas):

$$\Psi_{t_{nm}} = B^{(nm)} \partial_x^n \Psi, \qquad n, m \in \mathbb{N}_+, \tag{5}$$

where Ψ and w are square matrix functions and $B^{(nm)}, n, m \in \mathbb{N}_+$ are constant commuting square matrices. The integrability conditions yield the following hierarchy of C-integrable equations and its replicas:

$$w_{t_{nm}} + [w, B^{(nm)} W^{(n)}] - B^{(nm)} W_x^{(n)} = 0, \tag{6}$$

where

$$W^{(n)} = W_x^{(n-1)} + W^{(n-1)} w, \qquad n \in \mathbb{N}_+,$$
$$W^{(0)} = I, \quad W^{(1)} = w, \quad W^{(2)} = w_x + w^2, \quad W^{(3)} = w_{xx} + 2w_x w + w w_x + w^3, \dots \tag{7}$$

and I is the identity matrix.

The first three examples, together with their commuting replicas, read:

(1) $n = 1$: a C-integrable N-wave equation in (1+1) dimensions

$$w_{t_{1m}} - B^{(1m)} w_x + [w, B^{(1m)}] w = 0; \tag{8}$$

(2) $n = 2$: the matrix Bürgers equation

$$w_{t_{2m}} - B^{(2m)} w_{xx} - 2B^{(2m)} w_x w + [w, B^{(2m)}](w_x + w^2) = 0; \tag{9}$$

(3) $n = 3$: the third-order matrix Bürgers equation

$$w_{t_{3m}} - B^{(3m)} w_{xxx} - 3B^{(3m)} w_{xx} w + [w, B^{(3m)}](w_{xx} + w w_x + 2w_x w + w^3)$$
$$- 3B^{(3m)} w_x(w_x + w^2) = 0. \tag{10}$$

The way of generating solutions of the C-integrable PDEs (6) is elementary: take the general solution of equations (5):

$$\Psi(\vec{x}) = \int_\Gamma e^{Ikx + \sum_{j,m \geqslant 1} B^{(jm)} t_{jm} k^j} \hat{\Psi}(k) \, d\Omega(k), \tag{11}$$

where Γ is an arbitrary contour in the complex k-plane, $\Omega(k)$ is an arbitrary measure and $\hat{\Psi}(k)$ is an arbitrary matrix function of the spectral parameter k and $\vec{x}$ is the vector of all independent variables: $\vec{x} = \{x, t_{nm}; n, m \in \mathbb{N}_+\}$. Then

$$w = \Psi_x \Psi^{-1} \tag{12}$$

solves (6).

2.2. Block Frobenius matrix structure, integrable chains and S-integrable PDEs

It turns out that the C-integrable hierarchy of (1+1)-dimensional PDEs (6), including the N-wave, Bürgers and third-order Bürgers equations (8)–(10) as distinguished examples, generates a corresponding hierarchy of S-integrable (2+1)-dimensional PDEs, including the celebrated N-wave, DS and KP equations respectively. This is possible, due to the remarkable fact that equations (4) and (5) are compatible with the following block Frobenius matrix structure of the matrix function w:

$$w = \begin{bmatrix} w^{(1)} & w^{(2)} & w^{(3)} & \cdots \\ I_M & 0_M & 0_M & \cdots \\ 0_M & I_M & 0_M & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{13}$$

where I_M and 0_M are the $M \times M$ identity and zero matrices, $M \in \mathbb{N}_+$, and $w^{(j)}$, $j \in \mathbb{N}_+$ are $M \times M$ matrix functions. This block structure of w is consistent with equations (4) and (5) (and therefore with the whole C-integrable hierarchy (6)) iff matrix Ψ is a block Wronskian matrix:

$$\Psi = \begin{bmatrix} \Psi^{(11)} & \Psi^{(12)} & \Psi^{(13)} & \cdots \\ \partial_x^{-1}\Psi^{(11)} & \partial_x^{-1}\Psi^{(12)} & \partial_x^{-1}\Psi^{(13)} & \cdots \\ \partial_x^{-2}\Psi^{(11)} & \partial_x^{-2}\Psi^{(12)} & \partial_x^{-2}\Psi^{(13)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{14}$$

and

$$B^{(im)} = \mathrm{diag}(\tilde{B}^{(im)}, \tilde{B}^{(im)}, \cdots), \tag{15}$$

where the blocks $\Psi^{(ij)}$, $i, j \in \mathbb{N}_+$ are $M \times M$ matrices, and $\tilde{B}^{(im)}$, $i \in \mathbb{N}_+$ are constant commuting $M \times M$ matrices. In equations (13)–(15), the matrices w, Ψ and $B^{(im)}$ are chosen to be $\infty \times \infty$ square matrices containing an infinite number of finite blocks; only in dealing with the construction of explicit solutions, it is convenient to consider a finite number of blocks.

Substituting expressions (13) and (15) into the nonlinear PDEs (8)–(10), one obtains the following (by construction) integrable infinite chains of PDEs, for $n, m \in \mathbb{N}_+$:

$$w_{t_{1m}}^{(n)} - \tilde{B}^{(1m)} w_x^{(n)} + [w^{(n+1)}, \tilde{B}^{(1m)}] + [w^{(1)}, \tilde{B}^{(1m)}]w^{(n)} = 0, \tag{16}$$

$$w_{t_{2m}}^{(n)} - \tilde{B}^{(2m)} w_{xx}^{(n)} - 2\tilde{B}^{(2m)} w_x^{(n+1)} - 2\tilde{B}^{(2m)} w_x^{(1)} w^{(n)} + [w^{(1)}, \tilde{B}^{(2m)}]$$
$$\times \left(w^{(1)} w^{(n)} + w_x^{(n)} + w^{(n+1)}\right) + [w^{(2)}, \tilde{B}^{(2m)}]w^{(n)} + [w^{(n+2)}, \tilde{B}^{(2m)}] = 0, \tag{17}$$

$$w_{t_{3m}}^{(n)} - \tilde{B}^{(3m)}\left(w_{xxx}^{(n)} + 3\left(w_{xx}^{(1)} w^{(n)} + w_{xx}^{(n)}\right) + 3w_x^{(1)}\left(w^{(1)} w^{(n)} + w^{(n+1)} + w_x^{(n)}\right)\right.$$
$$\left.+ 3w_x^{(2)} w^{(n)} + 3w_x^{(n+2)}\right) + [w^{(1)}, \tilde{B}^{(3m)}]\left(w_{xx}^{(n)} + 2\left(w_x^{(1)} w^{(n)} + w_x^{(n+1)}\right)\right.$$
$$\left.+ w^{(1)}\left(w_x^{(n)} + w^{(1)} w^{(n)} + w^{(n+1)}\right) + w^{(2)} w^{(n)} + w^{(n+2)}\right) + [w^{(2)}, \tilde{B}^{(3m)}]$$
$$\times \left(w_x^{(n)} + w^{(1)} w^{(n)} + w^{(n+1)}\right) + [w^{(3)}, \tilde{B}^{(3m)}]w^{(n)} + [w^{(n+3)}, \tilde{B}^{(3m)}] = 0. \tag{18}$$

From these chains, whose spectral nature will be unveiled in section 2.4, one constructs, through a systematic elimination of some of the blocks $w^{(j)}$, the target S-integrable PDEs. Here we consider the following basic examples.

(2+1)-dimensional N-wave equation. Fixing $n = 1$ in equation (16), and choosing $m = 1, 2$, one obtains the following complete system of equations for $w^{(i)}$, $i = 1, 2$:

$$w_{t_{1m}}^{(1)} - \tilde{B}^{(1m)} w_x^{(1)} + [w^{(1)}, \tilde{B}^{(1m)}]w^{(1)} + [w^{(2)}, \tilde{B}^{(1m)}] = 0, \qquad m = 1, 2. \tag{19}$$

Eliminating $w^{(2)}$ from equations (19) one obtains the classical (2+1)-dimensional S-integrable N-wave equation:

$$\left[w_{t_{11}}^{(1)}, \tilde{B}^{(12)}\right] - \left[w_{t_{12}}^{(1)}, \tilde{B}^{(11)}\right] - \tilde{B}^{(11)} w_x^{(1)} \tilde{B}^{(12)} + \tilde{B}^{(12)} w_x^{(1)} \tilde{B}^{(11)}$$
$$+ [[w^{(1)}, \tilde{B}^{(11)}], [w^{(1)}, \tilde{B}^{(12)}]] = 0. \tag{20}$$

DS-type equation. Choosing $n = 1, 2$ in equations (16), $n = 1$ in equation (17), $m = 1$ in both equations, and simplifying the notation as follows:

$$t_j = t_{j1}, \qquad \tilde{B}^{(j)} = \tilde{B}^{(j1)}, \qquad j \in \mathbb{N}_+, \tag{21}$$

one obtains the following complete system of equations for $w^{(i)}$, $i = 1, 2, 3$:

$$w_{t_1}^{(1)} - \tilde{B}^{(1)} w_x^{(1)} + [w^{(2)}, \tilde{B}^{(1)}] + [w^{(1)}, \tilde{B}^{(1)}] w^{(1)} = 0, \tag{22}$$

$$w_{t_1}^{(2)} - \tilde{B}^{(1)} w_x^{(2)} + [w^{(3)}, \tilde{B}^{(1)}] + [w^{(1)}, \tilde{B}^{(1)}] w^{(2)} = 0, \tag{23}$$

$$w_{t_2}^{(1)} - \tilde{B}^{(2)} w_{xx}^{(1)} - 2\tilde{B}^{(2)} w_x^{(2)} - 2\tilde{B}^{(2)} w_x^{(1)} w^{(1)} + [w^{(1)}, \tilde{B}^{(2)}](w^{(1)} w^{(1)} + w_x^{(1)} + w^{(2)})$$
$$+ [w^{(2)}, \tilde{B}^{(2)}] w^{(1)} + [w^{(3)}, \tilde{B}^{(2)}] = 0. \tag{24}$$

Using equations (22) and (23), one can eliminate $w^{(3)}$ and $w^{(2)}$ from equation (24). In the case $\tilde{B}^{(2)} = \alpha \tilde{B}^{(1)}$ (α is a scalar), this results in the following equation for $w^{(1)}$:

$$[w_{t_2}^{(1)}, \tilde{B}^{(1)}] + \alpha(w_{t_1 t_1}^{(1)} - \tilde{B}^{(1)} w_{xx}^{(1)} \tilde{B}^{(1)} + [[w^{(1)}, \tilde{B}^{(1)}], w_{t_1}^{(1)}]$$
$$+ B^{(1)} w_x [B^{(1)}, w] - [B^{(1)}, w] w_x B^{(1)}) = 0. \tag{25}$$

In the simplest case of square matrices ($M = 2$), with $\tilde{B}^{(1)} = \beta \, \mathrm{diag}(1, -1)$ (β is a scalar constant), this equation reduces to the DS system:

$$\tilde{\beta} q_{t_2} - \frac{1}{2}\left(q_{xx} + \frac{1}{\beta^2} q_{t_1 t_1} \right) - 2(\varphi q + 2rq^2) = 0,$$

$$-\tilde{\beta} r_{t_2} - \frac{1}{2}\left(r_{xx} + \frac{1}{\beta^2} r_{t_1 t_1} \right) - 2(\varphi r + 2qr^2) = 0, \tag{26}$$

$$\varphi_{xx} - \frac{1}{\beta^2} \varphi_{t_1 t_1} + 4(rq)_{xx} = 0,$$

where

$$q = w_{12}^{(1)}, \qquad r = w_{21}^{(1)}, \qquad \varphi = (w_{11}^{(1)} + w_{22}^{(1)})_x, \qquad \tilde{\beta} = \frac{1}{\alpha \beta}. \tag{27}$$

If $\tilde{\beta} = \mathrm{i}$, this system admits the reduction $r = \bar{q}$:

$$\mathrm{i} q_{t_2} - \frac{1}{2}\left(q_{xx} + \frac{1}{\beta^2} q_{t_1 t_1} \right) - 2(\varphi q + 2\bar{q}q^2) = 0,$$

$$\varphi_{xx} - \frac{1}{\beta^2} \varphi_{t_1 t_1} + 4(\bar{q}q)_{xx} = 0, \tag{28}$$

becoming DS-I and DS-II if $\beta^2 = -1$ and $\beta^2 = 1$ respectively.

KP. To derive the celebrated KP equation, choose $M = 1$, take equations (17) with $n = 1, 2$, and equation (18) with $n = 1$, $\tilde{B}^{(2)} = \beta$, $\tilde{B}^{(3)} = -1$, where β is a scalar parameter, obtaining

$$w_{t_2}^{(1)} - \beta(w_{xx}^{(1)} + 2w^{(1)} w_x^{(1)} + 2w_x^{(2)}) = 0,$$
$$w_{t_2}^{(2)} - \beta(w_{xx}^{(2)} + 2w^{(2)} w_x^{(1)} + 2w_x^{(3)}) = 0,$$
$$w_{t_3}^{(1)} + w_{xxx}^{(1)} + 3((w^{(1)})^2 w_x^{(1)} + (w_x^{(1)})^2 + w^{(1)} w_{xx}^{(1)}) \tag{29}$$
$$+ 3(w_{xx}^{(2)} + w^{(2)} w_x^{(1)} + w^{(1)} w_x^{(2)}) + 3w_x^{(3)} = 0,$$

where we have set $m = 1$ and used again notations (21).

After eliminating $w^{(2)}$ and $w^{(3)}$, one obtains the scalar potential KP for $u = w^{(1)}$, $y = t_2$, $t = t_3$:

$$\left(u_t + \frac{1}{4} u_{xxx} + \frac{3}{2} u_x^2 \right)_x + \frac{3}{4\beta^2} u_{yy} = 0. \tag{30}$$

KP-I and KP-II correspond to $\beta^2 = -1$ and $\beta^2 = 1$ respectively.

2.3. Lax pairs for the S-integrable systems

Also the Lax pairs for the *S*-integrable systems derived in section 2.2 can be constructed in a similar way, from system (4), (5). We first observe that, due to equation (4), equations (5) can be rewritten as

$$\Psi_{t_{nm}} = B^{(nm)} W^{(n)} \Psi. \tag{31}$$

Due to the block Frobenius structure of w, it is convenient to work with the duals of equations (4) and (31):

$$\tilde{\Psi}_x = -\tilde{\Psi} w, \tag{32}$$

$$\tilde{\Psi}_{t_{nm}} = -\tilde{\Psi} B^{(nm)} W^{(n)}. \tag{33}$$

Substituting (13) and (15) into equations (32) and (33), one obtains a system of linear chains for the blocks of matrix $\tilde{\Psi}$. The first few equations involving the blocks of the first row read:

$$\tilde{\Psi}_x^{(1n)} + \tilde{\Psi}^{(11)} w^{(n)} + \tilde{\Psi}^{(1(n+1))} = 0, \tag{34}$$

$$\tilde{\Psi}_{t_{1m}}^{(1n)} + \tilde{\Psi}^{(11)} \tilde{B}^{(1m)} w^{(n)} + \tilde{\Psi}^{(1(n+1))} \tilde{B}^{(1m)} = 0, \tag{35}$$

$$\tilde{\Psi}_{t_{2m}}^{(1n)} + \tilde{\Psi}^{(11)} \tilde{B}^{(2m)} \left(w_x^{(n)} + w^{(1)} w^{(n)} + w^{(n+1)} \right) + \tilde{\Psi}^{(12)} \tilde{B}^{(2m)} w^{(n)} + \tilde{\Psi}^{(1(n+2))} \tilde{B}^{(2m)} = 0, \tag{36}$$

$$\tilde{\Psi}_{t_{3m}}^{(1n)} + \left(\tilde{\Psi}^{(11)} \tilde{B}^{(3m)} \left(w_{xx}^{(n)} + 2 \left(w_x^{(1)} w^{(n)} + w_x^{(n+1)} \right) + w^{(1)} \left(w_x^{(n)} + w^{(1)} w^{(n)} + w^{(n+1)} \right) \right. \right.$$
$$\left. + w^{(2)} w^{(n)} + w^{(n+2)} \right) + \tilde{\Psi}^{(12)} \tilde{B}^{(3m)} \left(w_x^{(n)} + w^{(1)} w^{(n)} + w^{(n+1)} \right)$$
$$+ \tilde{\Psi}^{(13)} \tilde{B}^{(3m)} w^{(n)} + \tilde{\Psi}^{(1(n+3))} \tilde{B}^{(3m)} = 0, \tag{37}$$

where $n \in \mathbb{N}_+$ and $\tilde{\Psi}^{(ij)}$ is the (i, j)-block of matrix $\tilde{\Psi}$.

Lax pair for the N-wave equation. Setting $n = 1$ into equations (34), (35) and eliminating $\tilde{\Psi}^{(12)}$ one obtains (the dual of) the Lax pair for the *N*-wave equation (20):

$$\tilde{\psi}_{t_{1m}} - \tilde{\psi}_x \tilde{B}^{(1m)} + \tilde{\psi} [\tilde{B}^{(1m)}, w^{(1)}] = 0, \qquad m = 1, 2, \tag{38}$$

where $\tilde{\psi} = \tilde{\Psi}^{(11)}$. The dual of it is the well-known Lax pair of the *N*-wave equation (20):

$$\psi_{t_{1m}} - \tilde{B}^{(1m)} \psi_x - [\tilde{B}^{(1m)}, w^{(1)}] \psi = 0, \qquad m = 1, 2. \tag{39}$$

Of course, the compatibility condition of equations (38) and/or equations (39) yields the nonlinear system (20).

Lax pair for DS. In this paragraph we set $m = 1$ in the integrable chains, and use notation (21). The first equation of the dual of the Lax pair is equation (38) with $m = 1$. To derive the second equation, we set $m = n = 1$ into equation (36), and eliminate the fields $\tilde{\Psi}^{(12)}$, $\tilde{\Psi}^{(13)}$, using equation (34) with $n = 1, 2$. In this way, one obtains the dual of the Lax pair for DS-type equations:

$$\tilde{\psi}_{t_1} - \tilde{\psi}_x \tilde{B}^{(1)} + \tilde{\psi} [\tilde{B}^{(1)}, w^{(1)}] = 0, \tag{40a}$$

$$\tilde{\psi}_{t_2} + \tilde{\psi}_{xx} \tilde{B}^{(2)} + \tilde{\psi}_x [w^{(1)}, \tilde{B}^{(2)}] + \tilde{\psi} \tilde{s} = 0, \tag{40b}$$

$$\tilde{s} = [\tilde{B}^{(2)}, w^{(2)}] + w_x^{(1)} \tilde{B}^{(2)} + \tilde{B}^{(2)} w_x^{(1)} + [\tilde{B}^{(2)}, w^{(1)}] w^{(1)}. \tag{40c}$$

Therefore the Lax pair reads:

$$\psi_{t_1} - \tilde{B}^{(1)} \psi_x - [\tilde{B}^{(1)}, w^{(1)}] \psi = 0, \tag{41a}$$

$$\psi_{t_2} - \tilde{B}^{(2)}\psi_{xx} + [w^{(1)}, \tilde{B}^{(2)}]\psi_x - s(y)\psi = 0, \tag{41b}$$

$$s = [\tilde{B}^{(2)}, w^{(2)}] + 2\tilde{B}^{(2)}w_x^{(1)} + [\tilde{B}^{(2)}, w^{(1)}]w^{(1)}. \tag{41c}$$

The compatibility conditions of equations (40) or (41) yield a nonlinear system equivalent to system (22)–(24).

Lax pair for KP. In this paragraph we use notations (21) as well. The first equations of the dual of the Lax pair for KP are the scalar versions of equations (40b) and (41b) respectively, with $\tilde{B}^{(2)} = \beta$ and $\tilde{B}^{(3)} = -1$. To write the second equation of the Lax pair for KP, we must take the scalar version of equation (37) with $m = n = 1$, and eliminate $\tilde{\Psi}^{(1i)}$, $i = 2, 3, 4$ using equations (36) for $n = 1, 2$. As a result, the dual of the Lax pair reads

$$\frac{1}{\beta}\tilde{\psi}_{t_2} + \tilde{\psi}_{xx} + 2\tilde{\psi}u_x = 0,$$
$$\tilde{\psi}_{t_3} + \tilde{\psi}_{xxx} + 3\tilde{\psi}_x u_x - \frac{3}{2}\tilde{\psi}\left(\frac{u_{t_2}}{\beta} - u_{xx}\right) = 0, \tag{42}$$

and the Lax pair is

$$\frac{1}{\beta}\psi_{t_2} - \psi_{xx} - 2u_x\psi = 0,$$
$$\psi_{t_3} + \psi_{xxx} + 3u_x\psi_x + \frac{3}{2}\left(\frac{u_{t_2}}{\beta} + u_{xx}\right)\psi = 0. \tag{43}$$

2.4. From the Lax pairs of S-integrable PDEs to C-integrable PDEs

As usual in the IST for (2+1)-dimensional soliton equations, one introduces the spectral parameter λ into the Lax pairs (39), (41) and (43) as follows:

$$\psi(\lambda; \vec{x}) = \chi(\lambda; \vec{x})\,e^{\lambda x I + \sum_{i,m \geqslant 1} \tilde{B}^{(im)} t_{im} \lambda^i}, \tag{44}$$

obtaining, respectively, the following spectral systems for the new eigenfunction χ:

$$\chi_{t_{1m}} - \tilde{B}^{(1m)}\chi_x - \lambda[\tilde{B}^{(1m)}, \chi] - [\tilde{B}^{(1m)}w^{(1)}]\chi = 0, \tag{45}$$

$$\chi_{t_1} - \tilde{B}^{(1)}\chi_x - \lambda[\tilde{B}^{(1)}, \chi] - [\tilde{B}^{(1)}w^{(1)}]\chi, \tag{46}$$

$$\chi_{t_2} - \tilde{B}^{(2)}\chi_{xx} + \lambda^2[\chi, \tilde{B}^{(2)}] - 2\lambda\tilde{B}^{(2)}\chi_x + [w^{(1)}, \tilde{B}^{(2)}](\chi_x + \lambda\chi) - s\chi = 0,$$
$$s = [\tilde{B}^{(2)}, w^{(2)}] + 2\tilde{B}^{(2)}w_x^{(1)} + [\tilde{B}^{(2)}, w^{(1)}]w^{(1)}. \tag{47}$$

$$\chi_{t_2} - \beta(2\lambda\chi_x + \chi_{xx} + 2\chi w_x) = 0,$$
$$\chi_{t_3} + 3\lambda\chi_{xx} + 3\lambda^2\chi_x + \chi_{xxx} + 3\lambda\chi w_x + \frac{3}{2}\chi\left(\frac{w_{t_2}}{\beta} + w_{xx}\right) = 0. \tag{48}$$

It is now easy to verify that the coefficients of the λ large expansion of the eigenfunction χ satisfy the infinite chains (16)–(18):

$$\chi(\lambda; \vec{x}) = I - \sum_{n \geqslant 1} \frac{w^{(n)}(\vec{x})}{\lambda^i}. \tag{49}$$

Therefore we have obtained the spectral interpretation of such chains. In addition, since the infinite chains (16)–(18) for $w^{(n)}$'s are equivalent, via the Frobenius structure (13), to C-integrable systems, we have also shown how to go backward, from S- to C-integrability.

2.5. Construction of solutions and Sato theory

In order to construct solutions of the *S*-integrable PDEs generated in section 2.2 from the elementary solution scheme (11), (12) of the matrix Bürgers hierarchy, we consider the matrices w and Ψ to be finite matrices consisting of $n_0 \times n_0$ blocks (this can be done assuming that $\Psi^{(1(n_0+1))} = 0$), where n_0 is an arbitrary positive integer greater than the number of blocks $w^{(j)}$'s involved in the *S*-integrable PDE under consideration. Taking into account the structures of w and Ψ given by equations (13) and (14) respectively, we have that

$$
w = \begin{bmatrix}
w^{(1)} & w^{(2)} & \cdots & \cdots & w^{(n_0)} \\
I_M & 0_M & \cdots & \cdots & 0_M \\
0_M & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0_M & I_M & 0_M
\end{bmatrix},
$$

$$
\Psi = \begin{bmatrix}
\Psi^{(11)} & \Psi^{(12)} & \cdots & \Psi^{(1n_0)} \\
\partial_x^{-1}\Psi^{(11)} & \partial_x^{-1}\Psi^{(12)} & \cdots & \partial_x^{-1}\Psi^{(1n_0)} \\
\vdots & \vdots & \vdots & \vdots \\
\partial_x^{-n_0+1}\Psi^{(11)} & \partial_x^{-n_0+1}\Psi^{(12)} & \cdots & \partial_x^{-n_0+1}\Psi^{(1n_0)}
\end{bmatrix}.
$$

(50)

We remark that the n_0 blocks $\Psi^{(1j)}$, $j = 1, \ldots, n_0$ of Ψ are defined, via (11), by equations

$$
\Psi^{(1j)}(\vec{x}) = \int e^{Ikx + \sum_{i,m \geqslant 1} \tilde{B}^{(im)} t_{im} k^i} \hat{\Psi}^{(1j)}(k)\, d\Omega(k), \qquad j = 1, \ldots, n_0 \quad (51)
$$

in terms of the arbitrary spectral functions $\hat{\Psi}^{(1j)}$, while the remaining blocks are constructed through the equations $\Psi^{(ij)} = \partial_x^{i-1}\Psi^{(1j)}$. Then, via (12), the components of the $M \times M$ blocks $w^{(i)}$ are expressed in terms of Ψ through the compact formula

$$
w_{\alpha\beta}^{(i)} = (\Psi_x \Psi^{-1})_{\alpha(iM-M+\beta)}, \qquad \alpha, \beta = 1, \ldots, M, \qquad i = 1, \ldots, n_0. \quad (52)
$$

This formula is intimately connected to those obtained via Sato theory. To see it, we consider the simplest case of scalar blocks ($M = 1$), containing the example of the KP equation. Then equation (52) becomes

$$
w^{(i)} = \frac{\triangle^{(i)}}{\triangle}, \tag{53}
$$

where

$$
\triangle = \begin{vmatrix}
\partial_x^{n_0-1} f^{(1)} & \partial_x^{n_0-1} f^{(2)} & \cdots & \partial_x^{n_0-1} f^{(n_0)} \\
\partial_x^{n_0-2} f^{(1)} & \partial_x^{n_0-2} f^{(2)} & \cdots & \partial_x^{n_0-2} f^{(n_0)} \\
\vdots & \vdots & \vdots & \vdots \\
f^{(1)} & f^{(2)} & \cdots & f^{(n_0)}
\end{vmatrix}, \tag{54}
$$

$$
\triangle^{(i)} = \begin{vmatrix}
\partial_x^{n_0-1} f^{(1)} & \partial_x^{n_0-1} f^{(2)} & \cdots & \cdots & \cdots & \cdots & \partial_x^{n_0-1} f^{(n_0)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\partial_x^{n_0-i} f^{(1)} & \partial_x^{n_0-i} f^{(2)} & \cdots & \cdots & \cdots & \cdots & \partial_x^{n_0-i} f^{(n_0)} \\
\partial_x^{n_0} f^{(1)} & \partial_x^{n_0} f^{(2)} & \cdots & \cdots & \cdots & \cdots & \partial_x^{n_0} f^{(n_0)} \\
\partial_x^{n_0-i-2} f^{(1)} & \partial_x^{n_0-i-2} f^{(2)} & \cdots & \cdots & \cdots & \cdots & \partial_x^{n_0-i-2} f^{(n_0)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
f^{(1)} & f^{(2)} & \cdots & \cdots & \cdots & \cdots & f^{(n_0)}
\end{vmatrix}, \tag{55}
$$

where $f^{(j)} = \partial_x^{-(n_0-1)}\Psi^{(1j)}$, $j = 1, \ldots, n_0$, equivalent to the formula obtained using Sato theory [41].

3. Relation between Ch- and S-integrability

Following the same strategy illustrated in section 2, in this section we establish the deep relations between the matrix PDE (1), recently introduced in [2] and integrated there by the method of characteristics, and (i) (1+1)-dimensional S-integrable soliton equations such as the KdV and NLS equations; (ii) the $GL(M, \mathbb{C})$ SDYM equation and its multidimensional generalizations; (iii) the multidimensional Calogero systems [42–46]. In this section, matrix w must be diagonalizable.

3.1. Matrix equations integrable by the method of characteristics

Consider the following matrix eigenvalue problem

$$w(\vec{x})\Psi(\Lambda; \vec{x}) = \Psi(\Lambda; \vec{x})\Lambda(\vec{x}), \tag{56}$$

for the matrix $w(\vec{x})$, where $\Lambda(\vec{x})$ is the diagonal matrix of eigenvalues, $\Psi(\Lambda; \vec{x})$ is a suitably normalized matrix of eigenvectors, and associate with it the following flows for $\Psi(\Lambda; \vec{x})$:

$$\Psi_{t_{mk}} + \sum_{j=1}^{N} \Psi_{x_{jk}}\rho^{(mjk)}(\Lambda) - B^{(mk)}\Psi\sigma^{(mk)}(\Lambda) = 0, \qquad m \in \mathbb{N}_+, \quad k = 1, 2, \tag{57}$$

where $B^{(mk)}$ are constant commuting matrices as in section 2 and $\vec{x}$ is the vector of all independent variables: $\vec{x} = (x_{jk}, t_{nm}, j, n, m \in \mathbb{N}_+, k = 1, 2)$. The compatibility between flows (57) implies the following commuting quasilinear PDEs for the eigenvalues:

$$\Lambda_{t_{mk}} + \sum_{j=1}^{N} \Lambda_{x_{jk}}\rho^{(mjk)}(\Lambda) = 0, \qquad m \in \mathbb{N}_+, \quad k = 1, 2; \tag{58}$$

the additional compatibility with the eigenvalue problem (56) implies the following nonlinear PDEs:

$$w_{t_{mk}} + \sum_{i=1}^{N} w_{x_{ik}}\rho^{(mik)}(w) + [w, B^{(mk)}]\sigma^{(mk)}(w) = 0, \qquad m \in \mathbb{N}_+, \quad k = 1, 2, \tag{59}$$

commuting replicas of equation (1).

   The way of solving equations (59) is as follows [2]. Consider the general solution of equations (58) with $k = 1, 2$, characterized by the following non-differential equations:

$$\Lambda = E\left(x_{11}I - \sum_{m\geqslant 1}\rho^{(m11)}(\Lambda)t_{m1}, \ldots, x_{N1}I - \sum_{m\geqslant 1}\rho^{(mN1)}(\Lambda)t_{m1};\right.$$

$$\left. x_{12}I - \sum_{m\geqslant 1}\rho^{(m12)}(\Lambda)t_{m2}, \ldots, x_{N2}I - \sum_{m\geqslant 1}\rho^{(mN2)}(\Lambda)t_{m2}\right), \tag{60}$$

where E is an arbitrary diagonal matrix function of $2N$ arguments; then the general solution of the linear matrix PDEs (57) for Ψ, with the convenient parametrization $\Psi_{ii} = 1$, is

given by

$$
\Psi_{\alpha\beta} = F_{\alpha\beta} \left(x_{11} - \sum_{m \geqslant 1} \rho^{(m11)}(\Lambda_\beta) t_{m1}, \ldots, x_{N1} - \sum_{m \geqslant 1} \rho^{(mN1)}(\Lambda_\beta) t_{m1}; \right.
$$

$$
\left. x_{12} I - \sum_{m \geqslant 1} \rho^{(m12)}(\Lambda_\beta) t_{m2}, \ldots, x_{N2} - \sum_{m \geqslant 1} \rho^{(mN2)}(\Lambda_\beta) t_{m2} \right)
$$

$$
\times \, \mathrm{e}^{\sum_{k=1}^{2} \sum_{m \geqslant 1} B_\alpha^{(mk)} \sigma^{(mk)}(\Lambda_\beta) t_{mk}}, \quad \alpha, \beta = 1, 2, \ldots \tag{61}
$$

where $F_{\alpha\beta}$ are arbitrary scalar functions of $2N$ arguments, with $F_{\alpha\alpha} = 1$. Then

$$
w = \Psi \Lambda \Psi^{-1} \tag{62}
$$

solves the nonlinear PDEs (59).

Now we proceed as in section 2, assuming for w the same block Frobenius matrix structure (13), consistent with equations (56) and (57) (and then with the hierarchies (59)) iff the matrices $B^{(mk)}$ are given as in (15), the diagonal matrix of eigenvalues $\Lambda(\vec{x})$ has the block structure

$$
\Lambda(\vec{x}) = \mathrm{diag}[\Lambda^{(1)}(\vec{x}), \Lambda^{(2)}(\vec{x}), \ldots], \tag{63}
$$

and

$$
\Psi = \begin{bmatrix}
\Psi^{(11)} & \Psi^{(22)}\Lambda^{(2)} & \Psi^{(33)}\Lambda^{(3)2} & \cdots \\
\Psi^{(11)}(\Lambda^{(1)})^{-1} & \Psi^{(22)} & \Psi^{(33)}\Lambda^{(3)} & \cdots \\
\Psi^{(11)}(\Lambda^{(1)})^{-2} & \Psi^{(22)}(\Lambda^{(2)})^{-1} & \Psi^{(33)} & \cdots \\
\Psi^{(11)}(\Lambda^{(1)})^{-3} & \Psi^{(22)}(\Lambda^{(2)})^{-2} & \Psi^{(33)}(\Lambda^{(3)})^{-1} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}. \tag{64}
$$

Using the same strategy as in section 2, we show that

(1) if $\rho^{(mjk)} = 0, k = 1, 2$, implying $\Lambda = $ const, equations (59) generate classical (1+1)-dimensional S-integrable PDEs such as the N-wave, NLS, KdV and mKdV equations;
(2) if $B^{(mk)} = 0$ (or $\sigma^{(mk)} = 0$), $k = 1, 2$, equations (59) generate the $GL(M, \mathbb{C})$ SDYM equation and its $(2N + 2)$-dimensional generalization;
(3) if $B^{(m1)} = 0$ (or $\sigma^{(m1)} = 0$) and $\rho^{(mi2)} = 0$, equations (59) generate Calogero systems.

3.2. Derivation of (1+1)-dimensional S-integrable PDEs

Let $\rho^{(mjk)} = 0$ in (57)–(59), implying $\Lambda = $ const (isospectral flows), and let $\sigma^{(mk)}(\Lambda) = \Lambda^m$, i.e.:

$$
w\Psi = \Psi\Lambda, \tag{65}
$$

$$
\Psi_{t_{mk}} - B^{(mk)}\Psi\Lambda^m = 0, \qquad m, k \in \mathbb{N}_+. \tag{66}
$$

The compatibility conditions for system (65), (66) yield, for $m \in \mathbb{N}_+$:

$$
w_{t_{mk}} + [w, B^{(mk)}]w^m = 0. \tag{67}
$$

We remark that these equations are equivalent to equations (6) with $\partial_x^j w = 0, \forall j$. Consequently, the discrete chains generated by equation (67), with $m = 1, k = 1, 2$ and $m = 2, 3, k = 1$, are given by equations (16)–(18) with $\partial_x^j w = 0, \forall j$:

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$$w_{t_{1k}}^{(n)} + [w^{(n+1)}, \tilde{B}^{(1k)}] + [w^{(1)}, \tilde{B}^{(1k)}]w^{(n)} = 0, \qquad k = 1, 2, \tag{68}$$

$$w_{t_2}^{(n)} + [w^{(1)}, \tilde{B}^{(2)}](w^{(1)}w^{(n)} + w^{(n+1)}) + [w^{(2)}, \tilde{B}^{(2)}]w^{(n)} + [w^{(n+2)}, \tilde{B}^{(2)}] = 0, \tag{69}$$

$$w_{t_3}^{(n)} + [w^{(1)}, \tilde{B}^{(3)}](w^{(1)}(w^{(1)}w^{(n)} + w^{(n+1)}) + w^{(2)}w^{(n)} + w^{(n+2)})$$
$$+ [w^{(2)}, \tilde{B}^{(3)}](w^{(1)}w^{(n)} + w^{(n+1)}) + [w^{(3)}, \tilde{B}^{(3)}]w^{(n)} + [w^{(n+3)}, \tilde{B}^{(3)}] = 0, \tag{70}$$

for $n \in \mathbb{N}_+$, where, in equations (69), (70), we have used the simplifying notation (21).

(1+1)-dimensional N-wave equation. Setting $n = 1$ into equations (68), and eliminating $w^{(2)}$, one obtains the well-known S-integrable N-wave system in (1+1) dimensions:

$$\left[w_{t_{11}}^{(1)}, \tilde{B}^{(12)}\right] - \left[w_{t_{12}}^{(1)}, \tilde{B}^{(11)}\right] + [[w^{(1)}, \tilde{B}^{(11)}], [w^{(1)}, \tilde{B}^{(12)}]] = 0. \tag{71}$$

NLS. Equation (68) with $k = 1$, $n = 1, 2$ and equation (69) with $n = 1$ are a complete system of PDEs for $w^{(j)}$, $j = 1, 2, 3$:

$$w_{t_1}^{(1)} + [w^{(2)}, \tilde{B}^{(1)}] + [w^{(1)}, \tilde{B}^{(1)}]w^{(1)} = 0,$$
$$w_{t_1}^{(2)} + [w^{(3)}, \tilde{B}^{(1)}] + [w^{(1)}, \tilde{B}^{(1)}]w^{(2)} = 0, \tag{72}$$
$$w_{t_2}^{(1)} + [w^{(1)}, \tilde{B}^{(2)}](w^{(1)}w^{(1)} + w^{(2)}) + [w^{(2)}, \tilde{B}^{(2)}]w^{(1)} + [w^{(3)}, \tilde{B}^{(2)}] = 0.$$

In the case $\tilde{B}^{(2)} = \alpha \tilde{B}^{(1)}$ (α is a scalar constant) this system results in the following equation for $w^{(1)}$:

$$\left[w_{t_2}^{(1)}, \tilde{B}^{(1)}\right] + w_{t_1 t_1}^{(1)} - \alpha\left[w_{t_1}^{(1)}w^{(1)}, \tilde{B}^{(1)}\right] + \alpha([w^{(1)}, \tilde{B}^{(1)}]w^{(1)})_{t_1} = 0. \tag{73}$$

If, in addition,

$$M = 2, \qquad \tilde{B}^{(1)} = \mathrm{diag}(1, -1), \tag{74}$$

this equation yields the celebrated NLS system:

$$\frac{1}{\alpha}q_{t_2} - \frac{1}{2}q_{\tau_1 \tau_1} - 4rq^2 = 0,$$
$$\frac{1}{\alpha}r_{t_2} + \frac{1}{2}r_{\tau_1 \tau_1} + 4qr^2 = 0 \tag{75}$$

for the off-diagonal elements of $w^{(1)}$: $q = w_{12}^{(1)}$, $r = w_{21}^{(1)}$. The NLS equation $iq_{t_2} + \frac{1}{2}q_{\tau_1 \tau_1} + 4q^2\bar{q} = 0$ corresponds to the reduction $r = \bar{q}$, $\alpha = i$.

KdV and mKdV. Equation (68) with $k = 1$, $n = 1, 2, 3$, and equation (70) with $n = 1$ yield a complete system of PDEs for $w^{(j)}$, $j = 1, 2, 3, 4$:

$$w_{t_1}^{(1)} + [w^{(2)}, \tilde{B}^{(1)}] + [w^{(1)}, \tilde{B}^{(1)}]w^{(1)} = 0,$$
$$w_{t_1}^{(2)} + [w^{(3)}, \tilde{B}^{(1)}] + [w^{(1)}, \tilde{B}^{(1)}]w^{(2)} = 0,$$
$$w_{t_1}^{(3)} + [w^{(4)}, \tilde{B}^{(1)}] + [w^{(1)}, \tilde{B}^{(1)}]w^{(3)} = 0, \tag{76}$$
$$w_{t_3}^{(1)} + [w^{(1)}, \tilde{B}^{(3)}](w^{(1)}(w^{(1)}w^{(1)} + w^{(2)}) + w^{(2)}w^{(1)} + w^{(3)})$$
$$+ [w^{(2)}, \tilde{B}^{(3)}](w^{(1)}w^{(1)} + w^{(2)}) + [w^{(3)}, \tilde{B}^{(3)}]w^{(1)} + [w^{(4)}, \tilde{B}^{(3)}] = 0.$$

In the case $\tilde{B}^{(3)} = -\tilde{B}^{(1)}$, this system reduces to the two-coupled matrix equations

$$w_{t_1}^{(1)} + [w^{(2)}, \tilde{B}^{(1)}] + [w^{(1)}, \tilde{B}^{(1)}]w^{(1)} = 0,$$
$$-\left[\tilde{B}^{(1)}, w_{t_1}^{(1)}\right] = w_{t_1 t_1}^{(2)} + \left[\tilde{B}^{(1)}, w_{t_1}^{(1)}(w^{(1)}w^{(1)} + w^{(2)}) + w_{t_1}^{(2)}w^{(1)}\right] - ([\tilde{B}^{(1)}, w^{(1)}]w^{(2)})_{t_1}. \tag{77}$$

If, in addition, choice (74) is made, system (77) becomes the mKdV system:

$$q_{t_3} + \frac{1}{4}q_{t_1 t_1 t_1} + 6q_{t_1}qr = 0,$$
$$r_{t_3} + \frac{1}{4}r_{t_1 t_1 t_1} + 6r_{t_1}rq = 0, \tag{78}$$

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where $q = w_{12}^{(1)}$, $r = w_{21}^{(1)}$, reducing to the KdV equation $q_{t_3} + \frac{1}{4}q_{t_1 t_1 t_1} + 6qq_{t_1} = 0$ and to the mKdV equation $q_{t_3} + \frac{1}{4}q_{t_1 t_1 t_1} + 6q^2 q_{t_1} = 0$ if $r = 1$ and $r = q$ respectively.

3.2.1. Lax pairs for S-integrable PDEs in (1+1) dimensions. As in section 2, in order to derive the Lax pairs for the above *S*-integrable PDEs in (1+1) dimensions, it is convenient to write system (65), (66) in the equivalent form

$$w\Psi = \Psi\Lambda, \tag{79}$$

$$\Psi_{t_{mk}} - B^{(mk)}w^m\Psi = 0, \qquad m \in \mathbb{N}_+, \tag{80}$$

and consider the dual system

$$\tilde{\Psi}w = \Lambda\tilde{\Psi}, \tag{81}$$

$$\tilde{\Psi}_{t_{mk}} + \tilde{\Psi}B^{(mk)}w^{(m)} = 0, \qquad m \in \mathbb{N}_+. \tag{82}$$

Taking into account the block structure of the matrix w given by equation (13) and considering the first rows of equations (81), (82), we obtain the following spectral chains, for $n \in \mathbb{N}_+$:

$$\tilde{\Psi}^{(1n)}\mathcal{E} = \tilde{\Psi}^{(11)}w^{(n)} + \tilde{\Psi}^{(1(n+1))}, \tag{83}$$

$$\tilde{\Psi}_{t_{1k}}^{(1n)} + \tilde{\Psi}^{(11)}\tilde{B}^{(1k)}w^{(n)} + \tilde{\Psi}^{(1(n+1))}\tilde{B}^{(1k)} = 0, \tag{84}$$

$$\tilde{\Psi}_{t_{2k}}^{(1n)} + \tilde{\Psi}^{(11)}\tilde{B}^{(2k)}(w^{(1)}w^{(n)} + w^{(n+1)}) + \tilde{\Psi}^{(12)}\tilde{B}^{(2k)}w^{(n)} + \tilde{\Psi}^{(1(n+2))}\tilde{B}^{(2k)} = 0, \tag{85}$$

$$\tilde{\Psi}_{t_{3k}}^{(1n)} + \tilde{\Psi}^{(11)}\tilde{B}^{(3k)}(w^{(1)}(w^{(1)}w^{(n)} + w^{(n+1)}) + w^{(2)}w^{(n)} + w^{(n+2)})$$
$$+ \tilde{\Psi}^{(12)}\tilde{B}^{(3k)}(w^{(1)}w^{(n)} + w^{(n+1)}) + \tilde{\Psi}^{(13)}\tilde{B}^{(3k)}w^{(n)} + \tilde{\Psi}^{(1(n+3))}\tilde{B}^{(3k)} = 0 \tag{86}$$

where $\mathcal{E} = \Lambda^{(1)}$. Setting $n = 1$ into equations (83), (84) and eliminating $\tilde{\Psi}^{(12)}$, one gets the dual of the Lax pair for the (1+1)-dimensional *N*-wave equation (71) ($\tilde{\psi} = \tilde{\Psi}^{(11)}$):

$$\tilde{\psi}_{t_{1k}} + \mathcal{E}\tilde{\psi}\tilde{B}^{(1k)} + \tilde{\psi}[\tilde{B}^{(1k)}, w^{(1)}] = 0, \qquad k = 1, 2. \tag{87}$$

Equation (87) with $k = 1$, written in terms of (21), is the first equation of the dual Lax pair also for equations (72) and (76).

Setting $k = n = 1$ into equation (85) and eliminating $\tilde{\Psi}^{(12)}$, $\tilde{\Psi}^{(13)}$ using equation (83), one gets the second equation of the dual Lax pair for (72):

$$\tilde{\psi}_{t_2} + \mathcal{E}^2\tilde{\psi}\tilde{B}^{(2)} + \mathcal{E}\tilde{\psi}[\tilde{B}^{(2)}, w^{(1)}] + \tilde{\psi}s = 0,$$
$$s = [\tilde{B}^{(2)}, w^{(2)}] + [\tilde{B}^{(2)}, w^{(1)}]w^{(1)}. \tag{88}$$

The second equation of the dual Lax pair for equation (76) results from equation (86), $k = n = 1$ after eliminating $\tilde{\Psi}^{(1j)}$, $j = 2, 3, 4$ using equation (83). In view of conditions (74) the complete dual spectral system reads:

$$\tilde{\psi}_{t_1} + \mathcal{E}\tilde{\psi}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \tilde{\psi}\begin{bmatrix} 0 & 2q \\ -2r & 0 \end{bmatrix} = 0,$$

$$\tilde{\psi}_{t_3} - \mathcal{E}^3\tilde{\psi}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \mathcal{E}^2\tilde{\psi}\begin{bmatrix} 0 & 2q \\ -2r & 0 \end{bmatrix}$$

$$+ \mathcal{E}\tilde{\psi}\begin{bmatrix} -2qr & -q_{\tau_1} \\ -r_{\tau_1} & 2qr \end{bmatrix} - \tilde{\psi}\begin{bmatrix} rq_{\tau_1} - qr_{\tau_1} & \frac{1}{2}q_{\tau_1\tau_1} + 4q^2r \\ -\frac{1}{2}r_{\tau_1\tau_1} - 4r^2q & qr_{\tau_1} - rq_{\tau_1} \end{bmatrix} = 0. \tag{89}$$

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The duals of equations (87)–(89) read:

$$\psi_{t_{1k}} - \tilde{B}^{(1k)}\psi\mathcal{E} - [\tilde{B}^{(1k)}, w^{(1)}]\psi = 0, \qquad k = 1, 2, \tag{90}$$

$$\psi_{t_2} - \tilde{B}^{(2)}\psi\mathcal{E}^2 - [\tilde{B}^{(2)}, w^{(1)}]\psi\mathcal{E} - s\psi = 0, \tag{91}$$

$$\psi_{t_1} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\psi\mathcal{E} - \begin{bmatrix} 0 & 2q \\ -2r & 0 \end{bmatrix}\psi = 0,$$

$$\psi_{t_3} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\psi\mathcal{E}^3 + \begin{bmatrix} 0 & 2q \\ -2r & 0 \end{bmatrix}\psi\mathcal{E}^2 \tag{92}$$

$$+ \begin{bmatrix} -2qr & -q_{\tau_1} \\ -r_{\tau_1} & 2qr \end{bmatrix}\psi\mathcal{E} + \begin{bmatrix} rq_{\tau_1} - qr_{\tau_1} & \frac{1}{2}q_{\tau_1\tau_1} + 4q^2 r \\ -\frac{1}{2}r_{\tau_1\tau_1} - 4r^2 q & qr_{\tau_1} - rq_{\tau_1} \end{bmatrix}\psi = 0.$$

Equations (90) are the Lax pair of the *N*-wave equation (71), equation (90) with $k = 1$ and equation (91) are the Lax pair of equation (73), reducing to the NLS system if (74) holds, and equations (92) are Lax pair of system (78) reducing to either KdV or mKdV.

3.2.2. *From the Lax pairs of (1+1)-dimensional S-integrable PDEs to the Ch-integrable equations (67).*

We show that equations (67) may be derived from the spectral problems obtained in section 3.2.1. Let

$$\psi(\Lambda; x) = \chi(\Lambda; x)\, e^{\sum_{i,k \geqslant 1}(-1)^i \tilde{B}^{(ik)} t_{ik}\Lambda^i}. \tag{93}$$

Then equations (90)–(92) yield:

$$\chi_{t_{1k}} + \Lambda[\tilde{B}^{(1k)}, \chi] - [\tilde{B}^{(1k)}, w^{(1)}]\chi = 0, \qquad k = 1, 2, \tag{94}$$

$$\chi_{t_2} - \Lambda^2[\tilde{B}^{(2)}, \chi] - [w^{(1)}, \tilde{B}^{(2)}]\Lambda\chi - s\chi = 0, \tag{95}$$

$$s = [\tilde{B}^{(2)}, w^{(2)}] + [\tilde{B}^{(2)}, w^{(1)}]w^{(1)}.$$

$$\chi_{t_1} + \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \chi\right]\Lambda - \begin{bmatrix} 0 & 2q \\ -2r & 0 \end{bmatrix}\chi = 0, \tag{96}$$

$$\chi_{t_3} - \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \chi\right]\Lambda^3 + \begin{bmatrix} 0 & 2q \\ -2r & 0 \end{bmatrix}\chi\Lambda^2$$

$$+ \begin{bmatrix} -2qr & -q_{\tau_1} \\ -r_{\tau_1} & 2qr \end{bmatrix}\chi\Lambda + \begin{bmatrix} rq_{\tau_1} - qr_{\tau_1} & \frac{1}{2}q_{\tau_1\tau_1} + 4q^2 r \\ -\frac{1}{2}r_{\tau_1\tau_1} - 4r^2 q & qr_{\tau_1} - rq_{\tau_1} \end{bmatrix}\chi. \tag{97}$$

It is now easy to verify, as in section 2, that the coefficients of the Λ large expansion of the eigenfunction χ satisfy the infinite chains (68)–(70):

$$\chi(\Lambda; y) \to I - \sum_{j \geqslant 1}(-1)^j w^{(j)}\Lambda^{-j}, \tag{98}$$

clarifying their spectral meaning. In addition, recompiling such coefficients into the block Frobenius matrix, we reconstruct the matrix equations (67) from the Lax pairs of (1+1)-dimensional *S*-integrable PDEs.

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3.3. Derivation of the SDYM equation and of its multi-dimensional generalizations

Now we take, in equations (56), (57), $B^{(nk)} = 0$ (or $\sigma^{(kj)} = 0$), $m = 1$, $\rho^{(1jk)}(\Lambda) = \Lambda^j$, $t_{11} = t$, $t_{12} = \tau$, $x_{j1} = x_j$, $x_{j2} = y_j$, obtaining the system

$$w\Psi = \Psi\Lambda,$$

$$\Psi_t + \sum_{j=1}^{N} \Psi_{x_j} \Lambda^j = 0, \tag{99}$$

$$\Psi_\tau + \sum_{j=1}^{N} \Psi_{y_j} \Lambda^j = 0,$$

whose compatibility condition yields

$$w_t + \sum_{j=1}^{N} w_{x_j} w^j = 0, \qquad w_\tau + \sum_{j=1}^{N} w_{y_j} w^j = 0. \tag{100}$$

We proceed as in the previous sections but, before considering the derivation in the general case, quite complicated, we illustrate the simplest two examples.

3.3.1. $N = 1$: the $GL(M, \mathbb{C})$ SDYM equation.
The compatibility condition of the system (99) yields

$$w_t + w_{x_1} w = 0, \qquad w_\tau + w_{y_1} w = 0. \tag{101}$$

Let w and Ψ be given by equations (13) and (64) respectively. The first rows of the matrix equations (101) generate the chains, for $n \in \mathbb{N}_+$ [37]:

$$w_t^{(n)} + w_{x_1}^{(1)} w^{(n)} + w_{x_1}^{(n+1)} = 0, \tag{102a}$$

$$w_\tau^{(n)} + w_{y_1}^{(1)} w^{(n)} + w_{y_1}^{(n+1)} = 0. \tag{102b}$$

Setting $n = 1$ and eliminating $w^{(2)}$, we derive the well-known $GL(M, \mathbb{C})$ SDYM equation:

$$w_{ty_1}^{(1)} - w_{\tau x_1}^{(1)} + \left[w_{x_1}^{(1)}, w_{y_1}^{(1)} \right] = 0. \tag{103}$$

To derive the Lax pair of (103), we first write the dual of system (99) in the convenient form:

$$\tilde{\Psi}w = \Lambda\tilde{\Psi}, \tag{104a}$$

$$\tilde{\Psi}_t + \tilde{\Psi}_{x_1} w = 0, \qquad \tilde{\Psi}_\tau + \tilde{\Psi}_{y_1} w = 0. \tag{104b}$$

Using again (13) and (64), the first rows of equations (104) appear in the form:

$$\tilde{\Psi}^{(11)} w^{(n)} + \tilde{\Psi}^{(1(n+1))} = \mathcal{E}\tilde{\Psi}^{(1n)},$$

$$\tilde{\Psi}_t^{(1n)} + \tilde{\Psi}_{x_1}^{(11)} w^{(n)} + \tilde{\Psi}_{x_1}^{(1(n+1))} = 0, \tag{105}$$

$$\tilde{\Psi}_\tau^{(1n)} + \tilde{\Psi}_{y_1}^{(11)} w^{(n)} + \tilde{\Psi}_{y_1}^{(1(n+1))} = 0,$$

where $\mathcal{E} = \Lambda^{(1)}$. Setting $n = 1$ into equations (105) and eliminating $\tilde{\Psi}^{(12)}$, one obtains the dual of the Lax pair of (103) for the spectral function $\tilde{\psi} = \tilde{\Psi}^{(11)}$:

$$\tilde{\psi}_t + (\mathcal{E}\tilde{\psi})_{x_1} = \tilde{\psi} w_{x_1}^{(1)}, \tag{106a}$$

$$\tilde{\psi}_\tau + (\mathcal{E}\tilde{\psi})_{y_1} = \tilde{\psi} w^{(1)}_{y_1}. \tag{106b}$$

Then the Lax pair of (103) reads:

$$\psi_t + \psi_{x_1}\mathcal{E} + w^{(1)}_{x_1}\psi = 0, \tag{107a}$$

$$\psi_\tau + \psi_{y_1}\mathcal{E} + w^{(1)}_{y_1}\psi = 0. \tag{107b}$$

Vice versa, it is easy to verify that the coefficients of the $\mathcal{E}$ large expansion of the eigenfunction ψ in (107) are the elements of chain (102):

$$\psi(\mathcal{E}; \vec{x}) \to I - \sum_{j \geqslant 1} w^{(j)}\mathcal{E}^{-j}, \tag{108}$$

obtaining the spectral meaning of such chains. As a consequence, one reconstructs equations (101) from the Lax pair (107) of the SDYM equation.

3.3.2. $N = 2$: a generalization of the SDYM equation in six dimensions.
The compatibility condition of system (99) yields now

$$w_t + w_{x_1}w + w_{x_2}w^2 = 0, \qquad w_\tau + w_{y_1}w + w_{y_2}w^2 = 0. \tag{109}$$

The corresponding chains read:

$$w^{(n)}_t + w^{(1)}_{x_1}w^{(n)} + w^{(n+1)}_{x_1} + w^{(1)}_{x_2}(w^{(1)}w^{(n)} + w^{(n+1)}) + w^{(2)}_{x_2}w^{(n)} + w^{(n+2)}_{x_2} = 0,$$
$$w^{(n)}_\tau + w^{(1)}_{y_1}w^{(n)} + w^{(n+1)}_{y_1} + w^{(1)}_{y_2}(w^{(1)}w^{(n)} + w^{(n+1)}) + w^{(2)}_{y_2}w^{(n)} + w^{(n+2)}_{y_2} = 0. \tag{110}$$

Setting $n = 1$ and $n = 2$ into (110) and eliminating the fields $w^{(3)}$, $w^{(4)}$, one obtains the following integrable system of two nonlinear PDEs in six dimensions for the fields $w^{(1)}$, $w^{(2)}$:

$$w^{(1)}_{x_2\tau} - w^{(2)}_{y_2 t} + w^{(2)}_{x_2 y_1} - w^{(2)}_{x_1 y_2} + w^{(1)}_{x_2 y_1}w^{(1)} - w^{(1)}_{x_1 y_2}w^{(1)} + w^{(1)}_{y_1}w^{(1)}_{x_2} - w^{(1)}_{x_1}w^{(1)}_{y_2} + [w^{(2)}_{y_2}, w^{(1)}_{x_2}]$$
$$+ [w^{(1)}_{y_2}, w^{(2)}_{x_2}] + w^{(1)}_{y_2}(w^{(1)^2})_{x_2} - w^{(1)}_{x_2}(w^{(1)^2})_{y_2} = 0,$$
$$w^{(1)}_{x_1\tau} - w^{(1)}_{y_1 t} + [w^{(1)}_{y_1}, w^{(1)}_{x_1}] + w^{(2)}_{x_2\tau} - w^{(2)}_{y_2 t} + [w^{(2)}_{y_2}, w^{(1)}_{x_1}] + [w^{(1)}_{y_1}, w^{(2)}_{x_2}] + [w^{(1)}_{x_2}, w^{(1)}_{y_2}]w^{(1)^2}$$
$$+ (w^{(1)}_{x_1 y_2}w^{(1)} - w^{(1)}_{x_2 y_1}w^{(1)} + w^{(2)}_{x_1 y_2} - w^{(2)}_{x_2 y_1})w^{(1)}$$
$$+ w^{(1)}_{x_2}(w^{(1)}_\tau - w^{(1)}w^{(1)}_{y_1} + [w^{(2)}_{y_2}, w^{(1)}])$$
$$- w^{(1)}_{y_2}(w^{(1)}_t - w^{(1)}w^{(1)}_{x_1} + [w^{(2)}_{x_2}, w^{(1)}]) + [w^{(2)}_{y_2}, w^{(2)}_{x_2}] = 0, \tag{111}$$

reducing to (103) for $w^{(1)}$ if the fields do not depend on x_2, y_2.

To derive the Lax pair of (111), we consider again the dual of system (99):

$$\tilde{\Psi}w = \Lambda\tilde{\Psi},$$
$$\tilde{\Psi}_t + \tilde{\Psi}_{x_1}w + \tilde{\Psi}_{x_2}w^2 = 0, \qquad \tilde{\Psi}_\tau + \tilde{\Psi}_{y_1}w + \tilde{\Psi}_{y_2}w^2 = 0. \tag{112}$$

Using (13) and (64), the first rows of equations (112) appear in the form:

$$\tilde{\Psi}^{(11)}w^{(n)} + \tilde{\Psi}^{(1(n+1))} = \mathcal{E}\tilde{\Psi}^{(1n)}, \tag{113}$$

$$\tilde{\Psi}^{(1n)}_t + \tilde{\Psi}^{(11)}_{x_1}w^{(n)} + \tilde{\Psi}^{(1(n+1))}_{x_1} + \tilde{\Psi}^{(11)}_{x_2}(w^{(1)}w^{(n)} + w^{(n+1)}) + \tilde{\Psi}^{(12)}_{x_2}w^{(n)} + \tilde{\Psi}^{(1(n+2))}_{x_2} = 0, \tag{114}$$

$$\tilde{\Psi}^{(1n)}_\tau + \tilde{\Psi}^{(11)}_{y_1}w^{(n)} + \tilde{\Psi}^{(1(n+1))}_{y_1} + \tilde{\Psi}^{(11)}_{y_2}(w^{(1)}w^{(n)} + w^{(n+1)}) + \tilde{\Psi}^{(12)}_{y_2}w^{(n)} + \tilde{\Psi}^{(1(n+2))}_{y_2} = 0. \tag{115}$$

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Setting $n = 1$ into equations (114), (115), and eliminating $\tilde{\Psi}^{(12)}$, $\tilde{\Psi}^{(13)}$ using equations (113) for $n = 1, 2$, one obtains the dual of the Lax pair of (111) for the spectral function $\tilde{\psi} = \tilde{\Psi}^{(11)}$:

$$\tilde{\psi}_t + \sum_{j=1}^{2} (\mathcal{E}^j \tilde{\psi})_{x_j} = \tilde{\psi} \left(w_{x_1}^{(1)} + w_{x_2}^{(2)} + w_{x_2}^{(1)} w^{(1)} \right) + \mathcal{E} \tilde{\psi} w_{x_2}^{(1)},$$

$$\tilde{\psi}_t + \sum_{j=1}^{2} (\mathcal{E}^j \tilde{\psi})_{y_j} = \tilde{\psi} \left(w_{y_1}^{(1)} + w_{y_2}^{(2)} + w_{y_2}^{(1)} w^{(1)} \right) + \mathcal{E} \tilde{\psi} w_{y_2}^{(1)}.$$

$$(116)$$

Then the Lax pair of (111) reads:

$$\psi_t + \psi_{x_1} \mathcal{E} + \psi_{x_2} \mathcal{E}^2 + \left(w_{x_2}^{(2)} + w_{x_2}^{(1)} w^{(1)} + w_{x_1}^{(1)} \right) \psi + w_{x_2}^{(1)} \psi \mathcal{E} = 0,$$

$$\psi_\tau + \psi_{y_1} \mathcal{E} + \psi_{y_2} \mathcal{E}^2 + \left(w_{y_2}^{(2)} + w_{y_2}^{(1)} w^{(1)} + w_{y_1}^{(1)} \right) \psi + w_{y_2}^{(1)} \psi \mathcal{E} = 0.$$

$$(117)$$

As before, it is easy to verify that equation (108) holds, namely that the coefficients of the $\mathcal{E}$ large expansion of ψ in (117) are the elements of chain (110). Therefore one reconstructs equations (109) from the Lax pair (111) of the six-dimensional generalization (111) of the SDYM equation.

3.3.3. Multidimensional generalization of the SDYM equation. Motivated by the above formulae for the simplest cases $N = 1, 2$, here we discuss the general N situation. If w is the block Frobenius matrix (13), the power w^j exhibits the following structure:

$$w^j = \begin{bmatrix} \tilde{w}^{(j;11)} & \tilde{w}^{(j;12)} & \tilde{w}^{(j;13)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{w}^{(j;j1)} & \tilde{w}^{(j;j2)} & \tilde{w}^{(j;j3)} & \cdots \\ I_M & 0_M & 0_M & \cdots \\ 0_M & I_M & 0_M & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad (118)$$

where the matrix blocks $\tilde{w}$ are defined by the equations

$$\tilde{w}^{(j;1n)} = \sum_{i=1}^{j-1} w^{(i)} \tilde{w}^{(j-i;1n)} + w^{(j+n-1)}, \qquad n, j \geqslant 1, \quad (\tilde{w}^{(1;1n)} = w^{(n)}), \qquad (119)$$

$$\tilde{w}^{(j;kn)} = \tilde{w}^{(j-k+1;1n)}, \qquad 2 \leqslant k \leqslant j, \quad (\tilde{w}^{(j;jn)} = w^{(n)}) \qquad (120)$$

in terms of the basic blocks $w^{(j)}$, $j \geqslant 1$. The first few examples read:

$$\tilde{w}^{(2;1n)} = w^{(1)} w^{(n)} + w^{(n+1)},$$
$$\tilde{w}^{(3;1n)} = (w^{(1)})^2 w^{(n)} + w^{(1)} w^{(n+1)} + w^{(2)} w^{(n)} + w^{(n+2)},$$
$$\tilde{w}^{(4;1n)} = w^{(1)} ((w^{(1)})^2 w^{(n)} + w^{(1)} w^{(n+1)} + w^{(2)} w^{(n)} + w^{(n+2)})$$
$$\qquad + w^{(2)} (w^{(1)} w^{(n)} + w^{(n+1)}) + w^{(3)} w^{(n)} + w^{(n+3)}. \qquad (121)$$

Furthermore, evaluating the $(1n)$-block of w^{j+1}, written as $(w^j w)$, we obtain the additional formula

$$\tilde{w}^{(j-1;1(n+1))} = \tilde{w}^{(j;1n)} - \tilde{w}^{(j-1;11)} w^{(n)}, \qquad j \geqslant 1, \quad n > 1, \qquad (122)$$

implying

$$\tilde{w}^{(j;1n)} = \tilde{w}^{(j+n-1;11)} - \sum_{l=1}^{n-1} \tilde{w}^{(j+l-1;11)} w^{(n-l)}, \qquad j, n \geqslant 1. \qquad (123)$$

Equation (123) reduces, for $j = 1$, to the following equation:

$$\tilde{w}^{(j;11)} = \sum_{i=1}^{j-1} \tilde{w}^{(j-i;11)} w^{(i)} + w^{(j)}, \qquad j \geqslant 1 \tag{124}$$

useful later on (compare it with equation (119) for $n = 1$).

Using equation (13), system (100) generates the discrete chains

$$w_t^{(n)} + \sum_{j=1}^{N} \left(\sum_{i=1}^{j} w_{x_j}^{(i)} \tilde{w}^{(j-i+1;1n)} + w_{x_j}^{(j+n)} \right) = 0,$$

$$w_\tau^{(n)} + \sum_{j=1}^{N} \left(\sum_{i=1}^{j} w_{y_j}^{(i)} \tilde{w}^{(j-i+1;in)} + w_{y_j}^{(j+n)} \right) = 0. \tag{125}$$

Setting $n = 1, \ldots, N$ into equations (125), one obtains a determined system of $2N$ equations for the fields $w^{(i)}, i = 1, \ldots, 2N$. As in the previous two illustrative examples for $N = 1, 2$, it is possible to eliminate the N fields $w^{(i)}, i = N + 1, \ldots, 2N$, obtaining a system on N equations in $(2N + 2)$ dimensions for the remaining fields $w^{(i)}, i = 1, \ldots, N$. Such system, which provides the natural multidimensional generalization of the SDYM equation (103), is conveniently written as follows:

$$p_\tau^{(N,0)} - q_t^{(N,0)} + [q^{(N;0)}, p^{(N;0)}] = 0,$$

$$p_\tau^{(N,n)} - q_t^{(N,n)} + \sum_{j=1}^{n} \left(p_{y_j}^{(N,n-j)} - q_{x_j}^{(N,n-j)} \right) + \sum_{j=0}^{n} [q^{(N,j)}, p^{(N,n-j)}] = 0, \tag{126}$$

$$1 \leqslant n \leqslant N - 1,$$

where the fields $p^{(N,j)}, q^{(N,j)}$ are suitable combinations of the N fields $w^{(n)}, n = 1, \ldots, N$:

$$p^{(N,j)} = \sum_{s=j+1}^{N} \left(\sum_{l=1}^{s-j-1} w_{x_s}^{(l)} \tilde{w}^{(s-j-l;11)} + w_{x_s}^{(s-j)} \right),$$

$$q^{(N,j)} = \sum_{s=j+1}^{N} \left(\sum_{l=1}^{s-j-1} w_{y_s}^{(l)} \tilde{w}^{(s-j-l;11)} + w_{y_s}^{(s-j)} \right); \tag{127}$$

the first few read as follows:

$$p^{(N,N-1)} = w_{x_N}^{(1)}, \qquad q^{(N,N-1)} = w_{y_N}^{(1)},$$

$$p^{(N,N-2)} = w_{x_{N-1}}^{(1)} + w_{x_N}^{(2)} + w_{x_N}^{(1)} w^{(1)}, \qquad q^{(N,N-2)} = w_{y_{N-1}}^{(1)} + w_{y_N}^{(2)} + w_{y_N}^{(1)} w^{(1)},$$

$$p^{(N,N-3)} = w_{x_{N-2}}^{(1)} + w_{x_{N-1}}^{(2)} + w_{x_N}^{(3)} + \left(w_{x_{N-1}}^{(1)} + w_{x_N}^{(2)} \right) w^{(1)} + w_{x_N}^{(1)} \left(w^{(1)2} + w^{(2)} \right), \tag{128}$$

$$q^{(N,N-3)} = w_{y_{N-2}}^{(1)} + w_{y_{N-1}}^{(2)} + w_{y_N}^{(3)} + \left(w_{y_{N-1}}^{(1)} + w_{y_N}^{(2)} \right) w^{(1)} + w_{y_N}^{(1)} \left(w^{(1)2} + w^{(2)} \right).$$

To show it, it is more convenient to go through the Lax pair derivation.

Lax pair. The system dual of (99) reads

$$\tilde{\Psi} w = \Lambda \tilde{\Psi},$$

$$\tilde{\Psi}_t + \sum_{j=1}^{N} \tilde{\Psi}_{x_j} w^j = 0, \qquad \tilde{\Psi}_\tau + \sum_{j=1}^{N} \tilde{\Psi}_{y_j} w^j = 0. \tag{129}$$

and it is conveniently rewritten in the equivalent form

$$\tilde{\Psi} w = \Lambda \tilde{\Psi},\tag{130}$$

$$\tilde{\Psi}_t + \sum_{j=1}^{N} (\Lambda^j \tilde{\Psi})_{x_j} - \sum_{j=1}^{N} \tilde{\Psi}(w^j)_{x_j} = \tilde{\Psi}_t + \sum_{j=1}^{N} (\Lambda^j \tilde{\Psi})_{x_j} - \sum_{s=0}^{N-1} \Lambda^s \tilde{\Psi} \sum_{j=s+1}^{N} w_{x_j} w^{j-s-1} = 0,\tag{131}$$

$$\tilde{\Psi}_\tau + \sum_{j=1}^{N} (\Lambda^j \tilde{\Psi})_{y_j} - \sum_{j=1}^{N} \tilde{\Psi}(w^j)_{y_j} = \tilde{\Psi}_t + \sum_{j=1}^{N} (\Lambda^j \tilde{\Psi})_{y_j} - \sum_{s=0}^{N-1} \Lambda^s \tilde{\Psi} \sum_{j=s+1}^{N} w_{y_j} w^{j-s-1} = 0.\tag{132}$$

As before, the block $(1, 1)$ of the matrix equations (131), (132) leads to the dual of the Lax pair $(\mathcal{E} = \Lambda^{(1)})$ of the multidimensional SDYM equations:

$$\tilde{\psi}_t + \sum_{j=1}^{N} (\mathcal{E}^j \tilde{\psi})_{x_j} = \sum_{j=0}^{N-1} \mathcal{E}^j \tilde{\psi} p^{(N,j)},\tag{133}$$

$$\tilde{\psi}_\tau + \sum_{j=1}^{N} (\mathcal{E}^j \tilde{\psi})_{y_j} = \sum_{j=0}^{N-1} \mathcal{E}^j \tilde{\psi} q^{(N,j)},\tag{134}$$

where $p^{(N,j)}$ and $q^{(N,j)}$ are defined in terms of $w^{(i)}$ and their derivatives in (127).

Then one derives the corresponding Lax pair

$$\psi_t + \sum_{j=1}^{N} \psi_{x_j} \mathcal{E}^j + \sum_{j=0}^{N-1} p^{(N,j)} \psi \mathcal{E}^j = 0,\tag{135}$$

$$\psi_\tau + \sum_{j=1}^{N} \psi_{y_j} \mathcal{E}^j + \sum_{j=0}^{N-1} q^{(N,j)} \psi \mathcal{E}^j = 0,\tag{136}$$

together with its compatibility condition, the following determined system of $2N$ equations in $(2N + 2)$ variables for the fields $p^{(N,j)}, q^{(N,j)}, j = 0, \ldots, N - 1$:

$$p_\tau^{(N,0)} - q_t^{(N,0)} + [q^{(N;0)}, p^{(N;0)}] = 0,\tag{137}$$

$$p_\tau^{(N,n)} - q_t^{(N,n)} + \sum_{j=1}^{n} \left(p_{y_j}^{(N,n-j)} - q_{x_j}^{(N,n-j)} \right) + \sum_{j=0}^{n} [q^{(N,j)}, p^{(N,n-j)}] = 0,$$

$$1 \leqslant n \leqslant N - 1,\tag{138}$$

$$\sum_{j=n-N+1}^{N} \left(p_{y_j}^{(N,n-j)} - q_{x_j}^{(N,n-j)} \right) + \sum_{j=n-N+1}^{N-1} [q^{(N,j)}, p^{(N,n-j)}] = 0, \quad N \leqslant n \leqslant 2N - 2,\tag{139}$$

$$p_{y_N}^{(N,N-1)} - q_{x_N}^{(N,N-1)} = 0.\tag{140}$$

We remark that only the first N equations (137), (138) involve derivatives with respect to the time variables t, τ; the remaining N equations (139), (140), providing a set of relations among the $2N$ fields $p^{(N,j)}, q^{(N,j)}$, are remarkably parametrized by equations (127) in terms of the N fields $w^{(j)}, j = 1, \ldots, N$. Therefore one is left with equations (126), (127).

We also remark that generalization (137)–(140) of the SDYM equation is known in the literature [18, 48], to be generated by the Lax pair

$$\psi_t + \sum_{j=1}^{N} \lambda^j \psi_{x_j} + \sum_{j=0}^{N-1} \lambda^j p^{(N,j)} \psi = 0, \tag{141}$$

$$\psi_\tau + \sum_{j=1}^{N} \lambda^j \psi_{y_j} + \sum_{j=0}^{N-1} \lambda^j q^{(N,j)} \psi = 0, \tag{142}$$

differing from (135), (136) by the fact that here λ is just a constant and scalar spectral parameter.

Therefore the remarkable derivation of (137)–(140) from the matrix equations (100) and its integration scheme has allowed one to uncover the following two important properties of system (137)–(140).

- Half of the equations of system (137)–(140) (the non-evolutionary part) can be parametrized in terms of the blocks $w^{(j)}$, $j = 1, \ldots, N$ of the Frobenius matrix w, reducing by half of the number of equations.
- Equations (137)–(140) turn out to be associated with the novel Lax pair (135), (136), in which the diagonal matrix $\mathcal{E}$ satisfies, from (58), the integrable quasilinear equations

$$\mathcal{E}_t + \sum_{j=1}^{N} \mathcal{E}_{x_j} \mathcal{E}^j = 0, \qquad \mathcal{E}_\tau + \sum_{j=1}^{N} \mathcal{E}_{y_j} \mathcal{E}^j = 0. \tag{143}$$

Therefore, as was already observed in [37] in the case of the SDYM equation (103), the integration scheme associated with such a novel Lax pair makes clear the existence of a rich solution space exhibiting interesting phenomena of multidimensional wave breaking. A detailed study of these solutions is postponed to a subsequent paper, together with the comparison with the finite gap solutions of the SDYM equation constructed in [49], and associated with a Riemann surface with branch points satisfying equations (143) for $N = 1$.

From the Lax pair of the multidimensional SDYM to the integrable chains (125). As for the particular cases $N = 1, 2$, in this section we show that the $\mathcal{E}$ large limit of the Lax pair (135), (136) yields expansion (108) for the eigenfunction ψ. Therefore the coefficients of the $\mathcal{E}$ large expansion of the spectral function associated with the *S*-integrable multidimensional generalization of the SDYM equations are solutions of the nonlinear chains (125), providing the spectral meaning to such nonlinear chains. In addition, recompiling the matrices $w^{(j)}$ into the block Frobenius matrix w, via (13), one establishes a remarkable relation between the Lax pair of the multidimensional SDYM and the basic matrix equations (100), solvable by the method of characteristics.

To show the validity of expansion (108), we substitute it into the Lax pair (135), (136), obtaining the following pairs of equations:

$$p^{(N,i)} = \sum_{j=i+1}^{N-1} \left(w_{x_j}^{(j-i)} + p^{(N,j)} w^{(j-i)} \right) + w_{x_N}^{(N-i)}, \qquad 0 \leqslant i \leqslant N - 1,$$

$$q^{(N,i)} = \sum_{j=i+1}^{N-1} \left(w_{y_j}^{(j-i)} + q^{(N,j)} w^{(j-i)} \right) + w_{y_N}^{(N-i)}, \qquad 0 \leqslant i \leqslant N - 1, \tag{144}$$

$$w_t^{(i)} + \sum_{j=1}^{N-1} \left(w_{x_j}^{(j+i)} + p^{(N,j)} w^{(j+i)} \right) + w_{x_N}^{(N+i)} + p^{(N;0)} w^{(i)} = 0, \qquad i \geqslant 1,$$

$$w_\tau^{(i)} + \sum_{j=1}^{N-1} \left(w_{y_j}^{(j+i)} + q^{(N,j)} w^{(j+i)} \right) + w_{y_N}^{(N+i)} + q^{(N;0)} w^{(i)} = 0, \qquad i \geqslant 1,$$

corresponding respectively to the condition that the coefficients of the positive and negative powers of $\mathcal{E}$ are zero in all orders. Equations (144) are identically satisfied using definitions (127) and (119) of $p^{(N,j)}$, $q^{(N,j)}$ and $\tilde{w}^{(j;11)}$. To show that also equations (145) are identically satisfied, we compare them with equations (125) and use again (127), (119), to finally derive the equation

$$\sum_{k=1}^{N} \sum_{l=1}^{k} w_{x_k}^{(l)} \left(\sum_{j=1}^{k-l} w^{(j)} \tilde{w}^{(k-l-j+1;1n)} - \sum_{j=0}^{k-l-1} \tilde{w}^{(k-l-j;11)} w^{(j+n)} \right) = 0, \qquad n \geqslant 1 \qquad (146)$$

identically satisfied, due to (119) and (124).

3.4. Derivation of Calogero systems

In the Lax pair of the *S*-integrable Calogero systems [42–46], the spectral problem is one-dimensional (like that of, say, KdV and NLS), while the equation describing the evolution of the eigenfunction is a multidimensional PDE in which an arbitrary number of additional independent variables are graded by powers of the spectral parameter; in addition, such spectral parameter satisfies a quasilinear PDE.

It is therefore clear that Calogero systems combine properties of the *S*-integrable PDEs in (1+1) dimensions of section 3.2 with properties of the multidimensional generalizations of the SDYM equation of section 3.3 and, to generate them from equations (59), we have to start with the matrix equation (56) and evolutionary system (57) in which $\rho^{(mj1)} = \sigma^{(m2)} = 0$:

$$w\Psi = \Psi\Lambda, \tag{147}$$

$$\Psi_{t_{m1}} - B^{(m1)} \Psi \sigma^{(m1)}(\Lambda) = 0, \qquad m \in \mathbb{N}_+,$$

$$\Psi_{t_{m2}} + \sum_{j=1}^{N} \Psi_{x_{j2}} \rho^{(mj2)}(\Lambda) = 0. \tag{148}$$

Here we illustrate the construction of the simplest example of the Calogero system, corresponding to $N = m = 1$ and $\sigma^{(11)}(\Lambda) = \rho^{(112)}(\Lambda) = \Lambda$, $B = B^{(11)}$, using the notation $\tilde{B} = \tilde{B}^{(11)}$, $\tau = t_{11}$, $t = t_{12}$, $x = x_{12}$. Then the compatibility between the two equations (148) yields the following quasilinear PDEs

$$\Lambda_t + \Lambda_x \Lambda = 0, \qquad \Lambda_\tau = 0 \tag{149}$$

for the matrix of eigenvalues; the compatibility between equations (147) and (148) yields instead the following matrix equations for field w:

$$w_\tau + [w, B]w = 0, \qquad w_t + w_x w = 0. \tag{150}$$

Using the Frobenius structure (13) of w, we get the following discrete chains, for $n \in \mathbb{N}_+$:

$$w_\tau^{(n)} + [w^{(n+1)}, \tilde{B}] + [w^{(1)}, \tilde{B}]w^{(n)} = 0,$$

$$w_t^{(n)} + w_x^{(1)} w^{(n)} + w_x^{(n+1)} = 0, \tag{151}$$

coinciding with equations (68) for $k = 1$, and with (102a). Fixing $n = 1$ in (151), and eliminating $w^{(2)}$, one gets the following matrix PDE:

$$\left[w^{(1)}_t, \tilde{B}\right] + \left[w^{(1)}_x w^{(1)}, \tilde{B}\right] = w^{(1)}_{\tau x} + ([w^{(1)}, \tilde{B}]w^{(1)})_x. \tag{152}$$

Let $M = 2$ and $\tilde{B} = \beta \operatorname{diag}(1, -1)$, then equation (152) yields

$$\begin{aligned}
-\beta q_t - \tfrac{1}{2}q_{x\tau} - 4\beta^2 q \partial_\tau^{-1}(qr)_x &= 0, \\
\beta r_t - \tfrac{1}{2}r_{x\tau} - 4\beta^2 r \partial_\tau^{-1}(qr)_x &= 0.
\end{aligned} \tag{153}$$

If $r = \bar{q}$, $\beta = \mathrm{i}$, then the above system becomes the following (2+1)-dimensional integrable variant of NLS:

$$\mathrm{i}q_t + \tfrac{1}{2}q_{xx} - 4q \partial_\tau^{-1}(q\bar{q})_x = 0, \tag{154}$$

studied in [50, 51].

Using the dual of equations (82) with $m = k = 1$, and equations (104a), (104b), we derive the dual Lax pair ((87) for $k = 1$ and (106a)) for (152):

$$\tilde{\psi}_\tau + \mathcal{E}\tilde{\psi}\tilde{B} + \tilde{\psi}[\tilde{B}, w^{(1)}] = 0, \qquad \tilde{\psi}_t + (\mathcal{E}\tilde{\psi})_x = \tilde{\psi}w^{(1)}_x \tag{155}$$

and the corresponding Lax pair ((90) for $k = 1$ and (107a)):

$$\psi_\tau - \tilde{B}\psi\mathcal{E} - [\tilde{B}, w^{(1)}]\psi = 0, \qquad \psi_t + \psi_x\mathcal{E} + w^{(1)}_x\psi = 0 \tag{156}$$

for system (152).

We end this section remarking that integrable PDEs associated with Lax pairs with varying spectral parameter have been studied also elsewhere, see, for instance, [52].

3.5. Construction of solutions

The construction of solutions for the three classes of *S*-integrable PDEs derived in sections 3.2, 3.3 and 3.4 from equation (59), is based on the solution of the algebraic equations (60), (61), taking into account the block-matrix structure (64) of Ψ. Then the independent blocks $\Psi^{(ii)}$, $i = 1, 2, \ldots$, are characterized by the explicit formula:

$$\begin{aligned}
\Psi^{(ii)}_{\alpha\beta} = F^{(ii)}_{\alpha\beta} &\left(x_{11} - \sum_{m\geqslant 1}\rho^{(m11)}\big(\Lambda^{(i)}_\beta\big)t_{m1}, \ldots, x_{N1} - \sum_{m\geqslant 1}\rho^{(mN1)}\big(\Lambda^{(i)}_\beta\big)t_{m1}; \right. \\
&\left. x_{12} - \sum_{m\geqslant 1}\rho^{(m12)}\big(\Lambda^{(i)}_\beta\big)t_{m2}, \ldots, x_{N2} - \sum_{m\geqslant 1}\rho^{(mN2)}\big(\Lambda^{(i)}_\beta\big)t_{m2} \right) \\
&\times \mathrm{e}^{\sum_{k=1}^2\sum_{m\geqslant 1}\tilde{B}^{(mk)}_\alpha\sigma^{(mk)}(\Lambda^{(i)}_\beta)t_{mk}}, \qquad \alpha, \beta = 1, \ldots, M, \quad i = 1, 2, \ldots, \quad (157)
\end{aligned}$$

where $F^{(ii)}_{\alpha\beta}$ are arbitrary scalar functions of $2N$ arguments (such that $F^{(ii)}_{\alpha\alpha} = 1$), while the remaining blocks are given by the equations $\Psi^{(ij)} = \Psi^{(jj)}(\Lambda^{(j)})^{j-i}$. Once Λ and Ψ are constructed in this way, the blocks $w^{(i)}$ are obtained, from equation (62), through the compact formula:

$$w^{(i)}_{\alpha\beta} = (\Psi\Lambda\Psi^{-1})_{\alpha(iM-M+\beta)}, \qquad \alpha, \beta = 1, \ldots, M. \tag{158}$$

In the case of the (1+1)-dimensional *S*-integrable models of section 3.2, corresponding to $\rho^{(ijk)} = 0$, Λ is an arbitrary constant diagonal matrix and formula (157) reduces to

$$\Psi^{(ii)}_{\alpha\beta} = F^{(ii)}_{\alpha\beta}\, \mathrm{e}^{\sum_{k=1}^2\sum_{m\geqslant 1}\tilde{B}^{(mk)}_\alpha\sigma^{(mk)}(\Lambda^{(i)}_\beta)t_{mk}}, \qquad \alpha, \beta = 1, \ldots, M, \quad i = 1, 2, \ldots, \tag{159}$$

where $F_{\alpha\beta}^{(ii)}$ are constant amplitudes.   Then solution (158) is a rational combination of exponentials. It would be interesting to compare, in this case, the solution space generated by (158) with that generated by Sato theory.

In the case of the Calogero systems of section 3.4, when $\rho^{(mj1)} = \sigma^{(m2)} = 0$, formula (157) reduces to

$$\Psi_{\alpha\beta}^{(ii)} = F_{\alpha\beta}^{(ii)} \left(x_{12} - \sum_{m \geqslant 1} \rho^{(m12)} \big(\Lambda_\beta^{(i)} \big) t_{m2}, \ldots, x_{N2} - \sum_{m \geqslant 1} \rho^{(mN2)} \big(\Lambda_\beta^{(i)} \big) t_{m2} \right)$$

$$\times e^{\sum_{m \geqslant 1} \tilde{B}_\alpha^{(m1)} \sigma^{(m1)} (\Lambda_\beta^{(i)}) t_{m1}}, \qquad \alpha, \beta = 1, \ldots, M, \quad i = 1, 2, \ldots. \qquad (160)$$

where $F_{\alpha\beta}^{(ii)}$ are now arbitrary functions of N arguments and Λ is the implicit solution of the nondifferential equation

$$\Lambda = E \left(x_{12} I - \sum_{m \geqslant 1} \rho^{(m12)} (\Lambda) t_{m2}, \ldots, x_{N2} I - \sum_{m \geqslant 1} \rho^{(mN2)} (\Lambda) t_{m2} \right), \quad (161)$$

following from equation (60), where E is the arbitrary diagonal matrix function of N arguments.

4. Generalizations

The block Frobenius matrix (13) is not the only possible structure of w allowing one to generate new types of integrable nonlinear PDEs, starting with C-integrable and Ch-integrable equations (6) and (59). A more general representation is given by the following block matrix:

$$w = \begin{bmatrix} W^{(11)} & W^{(12)} & \cdots \\ W^{(21)} & W^{(22)} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad (162)$$

where each block $W^{(ij)}$ has one of the following two block matrix forms:

$$W^{ij} = S^{(ij)} = \begin{bmatrix} w^{(ij;1)} & w^{(ij;2)} & w^{(ij;3)} & \cdots \\ I_M & 0_M & 0_M & \cdots \\ 0_M & I_M & 0_M & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$W^{ij} = C^{(ij)} = \begin{bmatrix} w^{(ij;1)} & w^{(ij;2)} & w^{(ij;3)} & \cdots \\ 0_M & I_M & 0_M & \cdots \\ 0_M & 0_M & I_M & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad (163)$$

and the blocks $w^{(ij;k)}$ are $M \times M$ matrices.

To provide consistency of this structure with the Lax pairs (4), (5) or (56), (57), we must take appropriate structure of matrices $B^{(nm)}$ and Ψ. Consider the simplest example of equations (4), (5) with $n = 1$. In this case w satisfies the N-wave equations (8). Let

$$w = \begin{bmatrix} S^{(11)} & C^{(12)} \\ C^{(21)} & C^{(22)} \end{bmatrix}. \qquad (164)$$

Then $B^{(1m)}$ must be taken in the form

$$B^{(1m)} = \text{diag}(\hat{B}^{(m1)}, \hat{B}^{(m2)}), \qquad \hat{B}^{(mj)} = \text{diag}(\tilde{B}^{(mj)}, \tilde{B}^{(mj)}, \ldots), \qquad j = 1, 2, \qquad (165)$$

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where $\tilde{B}^{(mj)}$ are diagonal matrices, and Ψ must have the following block structure:

$$\Psi = \begin{bmatrix} \Psi^{(11)} & \Psi^{(12)} \\ \Psi^{(21)} & \Psi^{(22)} \end{bmatrix}, \tag{166}$$

where $\Psi^{(ij)}$ must satisfy the following system of linear PDEs (consequence of equation (4)), for $i \geqslant 2$:

$$\Psi_x^{(11;ij)} = \Psi^{(11;(i-1)j)} + \Psi^{(21;ij)}, \tag{167}$$

$$\Psi_x^{(12;ij)} = \Psi^{(12;(i-1)j)} + \Psi^{(22;ij)}, \tag{168}$$

$$\Psi_x^{(21;ij)} = \Psi^{(11;ij)} + \Psi^{(21;ij)}, \tag{169}$$

$$\Psi_x^{(22;ij)} = \Psi^{(12;ij)} + \Psi^{(22;ij)}. \tag{170}$$

In view of (164), equation (8) reads ($t_i = t_{1i}$):

$$S_{t_i}^{(11)} - \hat{B}^{(i1)} S_x^{(11)} + [S^{(11)}, \hat{B}^{(i1)}]S^{(11)} + (C^{(12)}\hat{B}^{(i2)} - \hat{B}^{(i1)}C^{(12)})C^{(21)} = 0,$$
$$C_{t_i}^{(12)} - \hat{B}^{(i1)} C_x^{(12)} + [S^{(11)}, \hat{B}^{(i1)}]C^{(12)} + (C^{(12)}\hat{B}^{(i2)} - \hat{B}^{(i1)}C^{(12)})C^{(22)} = 0,$$
$$C_{t_i}^{(21)} - \hat{B}^{(i2)} C_x^{(21)} + (C^{(21)}\hat{B}^{(i1)} - \hat{B}^{(i2)}C^{(21)})S^{(11)} + [C^{(22)}, \hat{B}^{(i2)}]C^{(21)} = 0,$$
$$C_{t_i}^{(22)} - \hat{B}^{(i2)} C_x^{(22)} + (C^{(21)}\hat{B}^{(i1)} - \hat{B}^{(i2)}C^{(21)})C^{(12)} + [C^{(22)}, \hat{B}^{(i2)}]C^{(22)} = 0, \tag{171}$$

where $i = 1, 2$. Writing block (11) of each of these equations, one obtains

$$w_{t_i}^{(11;1)} - \tilde{B}^{(i1)} w_x^{(11;1)} + [w^{(11;1)}, \tilde{B}^{(i1)}]w^{(11;1)} + [w^{(11;2)}, \tilde{B}^{(i1)}]$$
$$+ (w^{(12;1)}\tilde{B}^{(i2)} - \tilde{B}^{(i1)}w^{(12;1)})w^{(21;1)} = 0, \tag{172a}$$

$$w_{t_i}^{(12;1)} - \tilde{B}^{(i1)} w_x^{(12;1)} + [w^{(11;1)}, \tilde{B}^{(i1)}]w^{(12;1)} + (w^{(12;1)}\tilde{B}^{(i2)} - \tilde{B}^{(i1)}w^{(12;1)})w^{(22;1)} = 0, \tag{172b}$$

$$w_{t_i}^{(21;1)} - \tilde{B}^{(i2)} w_x^{(21;1)} + (w^{(21;1)}\tilde{B}^{(i1)} - \tilde{B}^{(i2)}w^{(21;1)})w^{(11;1)}$$
$$+ (w^{(21;2)}\tilde{B}^{(i1)} - \tilde{B}^{(i2)}w^{(21;2)}) + [w^{(22;1)}, \tilde{B}^{(i2)}]w^{(21;1)} = 0, \tag{172c}$$

$$w_{t_i}^{(22;1)} - \tilde{B}^{(i2)} w_x^{(22;1)} + (w^{(21;1)}\tilde{B}^{(i1)} - \tilde{B}^{(i2)}w^{(21;1)})w^{(12;1)} + [w^{(22;1)}, \tilde{B}^{(i2)}]w^{(22;1)} = 0. \tag{172d}$$

Eliminating $w^{(11;2)}$ and $w^{(21;2)}$ from equations (172a), (172c) with $i = 1, 2$ and taking equations (172b), (172d) with $i = 2$, one obtains the following (2+1)-dimensional evolutionary system of PDEs in the time variable t_2:

$$[\mathcal{E}^{(1)}, \tilde{B}^{(21)}] - [\mathcal{E}^{(2)}, \tilde{B}^{(11)}] = 0, \tag{173}$$

$$q_{t_2} - \tilde{B}^{(21)}q_x + [v, \tilde{B}^{(21)}]q + (q\tilde{B}^{(22)} - \tilde{B}^{(21)}q)u = 0, \tag{174}$$

$$E^{(1)}\tilde{B}^{(21)} - \tilde{B}^{(22)}E^{(1)} - E^{(2)}\tilde{B}^{(11)} + \tilde{B}^{(12)}E^{(2)} = 0, \tag{175}$$

$$u_{t_2} - \tilde{B}^{(22)}u_x + (w\tilde{B}^{(21)} - \tilde{B}^{(22)}w)q + [u, \tilde{B}^{(22)}]u = 0, \tag{176}$$

where
$$\mathcal{E}^{(i)} = v_{t_i} - \tilde{B}^{(i1)}v_x + [v, \tilde{B}^{(i1)}]v + (q\tilde{B}^{(i2)} - \tilde{B}^{(i1)}q)w, \qquad i = 1, 2,$$
$$E^{(i)} = w_{t_i} - \tilde{B}^{(i2)}w_x + (w\tilde{B}^{(i1)} - \tilde{B}^{(i2)}w)v + [u, \tilde{B}^{(i2)}]w, \qquad i = 1, 2, \tag{177}$$

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and

$$v = w^{(11;1)}, \qquad q = w^{(12;1)}, \qquad w = w^{(21;1)}, \qquad u = w^{(22;1)}, \quad (178)$$

supplemented by equations (172*b*), (172*d*) with $i = 1$:

$$q_{t_1} - \tilde{B}^{(11)} q_x + [v, \tilde{B}^{(11)}]q + (q\tilde{B}^{(12)} - \tilde{B}^{(11)}q)u = 0, \qquad\qquad (179)$$

$$u_{t_1} - \tilde{B}^{(12)} u_x + (w\tilde{B}^{(11)} - \tilde{B}^{(12)}w)q + [u, \tilde{B}^{(12)}]u = 0, \qquad\qquad (180)$$

that can be considered as compatible constraints for the evolutionary system (173)–(176).

The Lax pair, as well as the solution space for this system, can be obtained following the procedures described in sections 2.3 and 2.5.

If $w = q = u = 0$, one obtains the *S*-integrable (2+1)-dimensional *N*-wave equation (173) for v (see equation (20)). Instead, if $v = w = q = 0$, one obtains the *C*-integrable (1+1)-dimensional *N*-wave-type equation (176) or (180) for u (see equation (8)). Therefore equations (173)–(176) and (179), (180) can be viewed as nonlinear PDEs sharing properties of *S*- and *C*-integrable systems. One can show that this property is shared by all the PDEs generated by reductions of the type (162), (163), recovering, in particular, the (*n*+1)-dimensional (*n* > 2) nonlinear PDEs constructed in [34] by a generalization of the dressing method.

5. Summary and future perspectives

We have established deep and remarkable connections among PDEs integrable by the inverse spectral transform method, the method of characteristics and the Hopf–Cole transformation. These relations can be used effectively to construct, for the generated *S*-integrable PDEs, the associated compatible systems of linear operators, their commuting flows and large classes of solutions. These results open several research perspectives.

(1) Use of the above derivation of the *S*-integrable systems to investigate the corresponding space of analytic solutions generated from the seed solutions of the original *C*- and *Ch*-integrable PDEs. In particular,

   (a) the connections between such solution space and that generated by Sato theory. In the KP case, the two solution spaces coincide; in other cases the connection is, at the moment, less clear.

   (b) The use of the quasilinear PDEs for the eigenvalues to study in detail the wave breaking phenomena associated with solutions of the SDYM equation and of its multidimensional generalizations.

(2) Search for the integrable systems that should generate, through a suitable matrix reduction, the integrable PDEs equivalent to the commutation of vector fields, like equation (3), a class of *S*-integrable systems not fitting yet into the general picture illustrated in this paper.

(3) Generalization of the techniques presented in this paper to generate novel integrable systems, possibly in multidimensions. In particular, a systematic use of group theory tools to explore reductions different from the block Frobenius matrix one.

(4) Construction of the discrete analogue of the results of this paper. In this respect, we remark that, while the discretization of the results of section 2 does not present, in principle, any conceptual problem, and will be the subject of a subsequent paper, the discretization of the results of section 3, if $\rho^{(ijk)} \neq 0$, is not clear, since a satisfactory discretization of the method of characteristics and of equation (1) for $\rho^{(i)} \neq 0$, in the scalar and matrix cases, are, at the moment, unknown.

Acknowledgments

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